Loudon's ALGEBRA FOR Beginners.

ALGEBRA FOR BEGINNERS.

BY

JAMES LOUDON, M.A.

PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY,
UNIVERSITY COLLEGE, TORONTO.

TORONTO:
COPP, CLARK & CO., FRONT STREET.
1876.

PREFACE.

Beginners, for whom this book is intended, are recommended to work according to the rules printed in italics, and to omit, on a first reading, the articles in small type. Those who desire only a practical acquaintance with the methods of working, may also omit Chapters X and XXI.

The author will feel obliged to teachers for any suggestions they may have to communicate.

SEPTEMBER, 1875.

CONTENTS.

		PAGE.
I.	Introduction	1
II.	Addition	8
III.	Subtraction	16
IV.	The use of Double Signs and Brackets	20
V.	Multiplication	23
VI.	Division	30
VII.	Examples involving the application of the first four Rules	39
VIII.	Simple Equations	42
IX.	Problems	49
X.	Particular results in Multiplication and Division	55
XI.	Involution and Evolution	61
XII.	The Highest Common Measure	72
XIII.	The Lowest Common Multiple	84
XIV.	Fractions	88
XV.	Simple Equations, continued	105
XVI.	Problems, continued	108
XVII.	Quadratic Equations	113
XVIII.	Problems	119
XIX.	Simultaneous Equations	123
XX.	Problems	129
XXI.	Exponential Notation	133
	Answers	141

ALGEBRA.

CHAPTER I.

INTRODUCTION.

1. The operations of addition and subtraction, which in Arithmetic are stated in words, are denoted in Algebra by the signs + and −, respectively.

Thus, *add together* 12, 5, *and* 6, is expressed $12+5+6$; *from* 15 *take* 9, is expressed $15-9$; *add together* 3, $\frac{1}{2}$, $5\frac{1}{6}$, *and from the sum take* $\frac{1}{4}$ *and* $3\frac{1}{5}$, is expressed $3+\frac{1}{2}+5\frac{1}{6}-\frac{1}{4}-3\frac{1}{5}$; and so on.

2. The sign + is called the *plus* sign, and the sign − the *minus* sign.

Thus, $24+2-15$, is read 24 *plus* 2 *minus* 15.

Exercise I.

Employ *plus* and *minus* signs to express the following operations in Algebraical language:—

(1.) Add together 56, 10 and 15; 10, 12 and $\frac{4}{5}$; 2, $\frac{3}{4}$ and $\frac{6}{7}$; $\frac{5}{6}$, $\frac{9}{10}$, $\frac{11}{12}$, and $\frac{1}{3}$.

(2.) From 29 take 15; from $2\frac{1}{3}$ take $1\frac{1}{2}$; from 2·5 take 1·6.

(3.) From the sum of 11, 35 and 6, take 17; from the sum of 81 and 75 take 69 and 42.

(4.) To the difference between 15 and 7 add the sum of 8 and 9.

(5.) To the difference between 28 and 16 add the difference between 10 and 4.

3. The signs + and − are also used to denote that the quantities before which they are written are, respectively, *to be added to* and *subtracted from* some quantity not necessarily expressed.

Thus, $+4$ denotes a number 4 *to be added to* some number; -5 denotes a number 5 *to be subtracted from* some number.

4. Quantities to be added are called *positive* quantities, and quantities to be subtracted *negative* quantities.

Thus, $+2$, $+7$, $+\frac{1}{2}$, are positive quantities; and -3, -1, $-\frac{4}{5}$, are negative quantities.

5. As an illustration of positive and negative quantities, we may take the following examples:—

(i.) A man has a certain amount of cash in hand; he owes \$150 to one man, and \$280 to another; and there are owing to him the several sums of \$100, \$210, and \$120. Now, in order to determine what the man is worth, these several sums are to be considered in connection with the cash in hand; \$150 and \$280 are evidently *to be subtracted*, and \$100, \$210, and \$120, *to be added.* The former, therefore, may be denoted by -150, -280, and the latter by $+100$, $+210$, $+120$. In other words, in the process of finding out how a man's business stands, the sign + may be used to denote his assets, and the sign − his liabilities.

(ii.) The mercury in a thermometer rises and falls in consequence of changes in the temperature, and the amounts of these variations are expressed in degrees. In order to determine the reading of the thermometer, some of these variations must be added to, and others subtracted from, the reading before the variations took place. Suppose, for example, that during the day there take place a rise of 5°, a fall of 7°, a

rise of 12°, and a fall of 8°; 5° and 12° are to be added to, and 7° and 8° subtracted from, the reading of the morning. The former, therefore, may be denoted by $+5°, +12°$, and the latter by $-7°$, $-8°$. If the degrees are measured from a zero point, distances above may therefore be denoted by $+$, and distances below by $-$. Thus $+20°$ means 20° above zero, and $-5°$ means 5° below zero.

6. In like manner the signs $+$ and $-$ may generally be employed to denote the two relations of contrariety which magnitudes of the same kind may bear to one another in some defined respect. Thus, if money received be denoted by $+$, money paid away will be denoted by $-$; if distances walked in one direction be denoted by $+$, distances walked in the contrary direction will be denoted by $-$; if $+$ denote games won, $-$ will denote games lost; and so on.

Exercise II.

(1.) A owes B $60, B owes C $30, and C owes A $20; employ the signs $+$ and $-$ to denote the assets and liabilities of A, B, C.

(2.) B and C owe A $20 each, C and A owe B $30 each, and A and B owe C $40 each; express in Algebraical language the assets and liabilities of A, B, C.

(3.) A pays B $10, B pays C $7, and C pays A $4; express Algebraically the amounts paid and received by A, B, C.

(4.) Denote the following variations in the thermometer: a fall of 2°, a rise of 5°, a fall of 3°.

(5.) Denote that the thermometer rises and falls alternately 1° per hour for 5 hours.

(6.) Denote that the thermometer falls and rises alternately 2° per hour for 4 hours.

(7.) Denote the following readings: 25° above zero, 7° below zero.

7. In Algebra the numerical values of quantities are denoted

by (1) figures, as in Arithmetic; (2) the letters of the alphabet, either alone or in combination.

Thus, if there be three points, A, B, C, situated in that order on a right line, the distances between A and B, and between B and C, if unknown or variable, may be denoted by a and b feet, respectively; and, consequently, the distance between A and C will be denoted by $a+b$ feet. If C lie between A and B, and a and b denote, as before, the lengths AB, BC, the length AC will be denoted by $a-b$ feet.

Again, if x denote the length, and y the breadth, in feet, of a room, the dimensions of a room 3 feet longer and 5 feet narrower will be $x+3$ and $y-5$ feet, respectively.

8. In Arithmetical operations which are performed with figure symbols alone, all mention of the unit is suppressed, the results of these operations being true, whatever be the unit in view. Thus, 10 and 5 added together make 15, whether the suppressed unit be a pound, a gallon, or an inch. Algebraical symbols have a still greater generality. Not only is the unit suppressed, but the number of units is not assigned, as in Arithmetic. Thus a may represent any number referred to any unit. Operations performed in Algebraical symbols will, therefore, give results which are true for any numerical values which may be assigned to the symbols.

9. The product of symbols which denote numbers, is represented by writing them down in a horizontal line one after another, in any order, with or without the multiplication sign $\times$, or dot $\cdot$, between them.

Thus ab, ba, $a \cdot b$, $b \cdot a$ $a \times b$, $b \times a$, all denote the product of a and b; abc, $a \cdot b \cdot c$, $a \times b \times c$, the product of a, b, c; and so on.

Figure symbols are written first in order; thus, $3a$, $5ab$, $6abc$, $\frac{2}{3}a$. When there are two figure symbols, the sign $\times$ only must be used between them.

Thus, the product of 5 and $6a$ is denoted by $5 \times 6a$, and not by $5 \cdot 6a$, or $56a$, whose values are *five decimal six times a*, and *fifty-six times a*, respectively.

When there are three or more figure symbols, either the sign $\times$ or $\cdot$ must be used.

Thus the product of 2, 5 and 7 is denoted by $2 \cdot 5 \cdot 7$, or $2 \times 5 \times 7$.

10. The symbol a^2 stands for aa; a^3 for aaa; a^4 for $aaaa$; and so on. a^2 is read *a squared*, or *a to the power of* 2; a^3 is read *a cubed*, or *a to the power of* 3; a^4 is read *a to the power of* 4; and so on.

11. The power to which a letter is raised is called the *index* or *exponent* of that power.

Thus, 1 is the index or exponent of a; 2 of a^2; 3 of a^3; 4 of a^4.

12. The quotient of one quantity, a, divided by another, b, is denoted by either of the forms, $a \div b$ or $\frac{a}{b}$.

Thus the quotient of $2a$ divided by $3bc$ is denoted by $2a \div 3bc$ or by $\frac{2a}{3bc}$.

13. Any object or result of an Algebraical operation is called an Algebraical *quantity* or *expression*.

14. It is to be observed that the numerical value of an expression which is not fractional in form may be a fraction, and the numerical value of an expression which is fractional in form may be an integer.

Thus, if to a we assign the value 2, and to b the value $\frac{1}{2}$, the value of $\frac{a}{b}$ will be $2 \div \frac{1}{2}$, or 4; whilst if the values of a and b be $\frac{1}{2}$ and $\frac{1}{4}$, respectively, the value of ab will be $\frac{1}{2} \times \frac{1}{4}$, or $\frac{1}{8}$, and the value of $\frac{a}{b}$ will be $\frac{1}{2} \div \frac{1}{4}$, or 2.

15. The symbol $=$ stands for *is* or *are equal to*, and is written between the quantities whose equality it is desired to express.

Thus $a = 3$ denotes that the value of a is 3.

Exercise III.

If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $x=0$, find the values of the following expressions:—

(1.) $a+2b+3c+4d$. (2.) $13a-4b+5c$. (3.) $ab+de$.

(4.) $ab+de-cx$. (5.) $4abc$, $5bcd$, $6dex$.

(6.) $a^2+b^2+c^2$. (7.) $4a^2+3b^2-c^2$. (8.) $5ad+4b^3-c^2d$.

(9.) $\frac{a}{b}+\frac{d}{e}$. (10.) $\frac{c}{d}-\frac{a}{b}+\frac{x}{e}$. (11.) $\frac{ab}{de}+\frac{ad}{bc}-\frac{bx}{ce}$.

(12.) $\frac{a^3}{5b^2}-\frac{2b}{3c^2d}+\frac{3d^2e}{8a^2b^4}$.

(13.) If $b=c=d=a$, find what each of the following becomes in terms of a:

$$2bca,\ 3a^2b,\ 2a^2+3abc+4b^2cd.$$

16. When two or more quantities are multiplied together, each is said to be a factor of the product.

Thus, a and b are factors of ab; 3 and a^2 are factors of $3a^2$; and 5, b, and c are factors of $5bc$.

17. One factor of a quantity is said to be a coefficient of the remaining factor, and is said to be a *literal* or a *numerical* coefficient according as it involves letters or not.

Thus, in $3x$ and $\frac{4}{5}a^2$ the numerical coefficients of x and a^2 are 3 and $\frac{4}{5}$, respectively; in ax^2 and $3cd$ the literal coefficients of x^2 and d are a and $3c$, respectively.

Also, since $x=1\times x$, the coefficient of x in the quantity x is 1.

The sign $+$ or $-$ when it precedes a quantity is also a sign of the coefficient.

Thus, the coefficient of x in $+3x$ is $+3$, of x^2 in $-5ax^2$ is $-5a$, and of dz^2 in $-\frac{2}{3}cdz^2$ is $-\frac{2}{3}c$.

In the case of fractional numerical coefficients the letter symbols are sometimes written with the numerator; thus,

$$\frac{a}{2} = \tfrac{1}{2}a, \frac{2x}{5} = \tfrac{2}{5}x, -\frac{6a^2b}{7} = -\tfrac{6}{7}a^2b.$$

18. Quantities are said to be *like* or *unlike* according as they involve the same or different combinations of letters.

Thus, $+5a$, $-7a$ are like quantities; and so also are $+6x^2y$, $-5x^2y$; $-2a$, $-a^2$ are unlike quantities.

Exercise IV.

(1.) Name the coefficient of x in $2x$, $3ax$, $\tfrac{4}{5}x$, bcx, $4a^2bx$, $\tfrac{1}{2}ax$.

(2.) Name the coefficient of a in $-a$, $+3a$, $-\tfrac{5}{7}a$, $+2ab$, $-5ax^2$.

(3.) Name the coefficient of x^2 in $+x^2$, $-x^2$, $-3ax^2$, $+\tfrac{2}{3}cdx^2$.

(4.) Name the coefficient of xy in $-xy$, $+3a^2xy$, $-\tfrac{2}{3}axyz$.

(5.) Name the numerical coefficients in $\frac{a}{2}$, $\frac{2x}{5}$, $-\frac{x}{4}$, $+\frac{3x}{7}$, $-\frac{5x}{6}$.

(6.) Name the numerical coefficients in x, $-y$, $-2x^2$, $-3yz$, $+\tfrac{1}{2}a$, $-\tfrac{3}{4}bc$.

(7.) Name the like quantities among $2x$, ax, x, $3cx^2$.

(8.) Name the like quantities among $-a^2$, $+2a^2x$, $+\tfrac{2}{3}a^2$, $-a^2x$.

(9.) Name the like quantities among $-3a^2x$, ax^2, abx^2, $+2a^2x$, $-ax^2$.

CHAPTER II.

ADDITION.

I. Figure Symbols.

19. In order to explain the meaning of Algebraical *addition*, we shall, in the first instance, suppose the numerical values of the quantities to be represented by figure symbols, as in Arithmetic. We shall consider in order the addition of

(i.) Positive Quantities.

(ii.) Negative Quantities.

(iii.) Positive and Negative Quantities.

20. (i.) Positive Quantities.

We have seen that a positive quantity, as $+5$, represents a quantity 5 *to be added to* some number which may or may not be known or expressed. *The sum of any number of positive quantities is denoted by writing them in a row with their signs between them, or by a positive quantity whose numerical value is their Arithmetical sum.*

Thus the sum of $+4$, $+2$, and $+10$ is $+4+2+10$, or $+16$; that is to say, the addition of 4 and 2 and 10 to a number is equivalent to the addition of 16 to that number.

In like manner, the sum of $+2$, $+\frac{2}{3}$, $+\frac{4}{5}$ and $+5$ is

$$+2+\tfrac{2}{3}+\tfrac{4}{5}+5=+8\tfrac{7}{15}.$$

The Algebraical statement

$$+5+6+\tfrac{1}{3}+3\tfrac{1}{3}=+14\tfrac{2}{3}$$

may therefore be read *the addition of* 5, 6, $\frac{1}{3}$ *and* $3\frac{1}{3}$ *to a number is equivalent to the addition of* $14\frac{2}{3}$ *to that number.*

21. (ii.) Negative Quantities.

Since -4 denotes a quantity 4 *to be subtracted from* some number, when there are several negative quantities, as -4, -7, -8, denoting that they are all to be subtracted from some number, the operation may be denoted by writing them in a row, thus $-4-7-8$, or by a negative quantity, -19, whose numerical value is their Arithmetical sum; that is to say,

$$-4-7-8=-19.$$

In Algebra $-4-7-8$, or -19, is called *the sum of* -4, -7, *and* -8. Thus the sum of -1, -10, $-\frac{2}{3}$, and $-\frac{6}{7}$ is

$$-1-10-\tfrac{2}{3}-\tfrac{6}{7}=-12\tfrac{11}{21}.$$

It follows therefore that *the Algebraical sum of any number of negative quantities is a negative quantity whose numerical value is their Arithmetical sum.*

It will be observed that, instead of saying *to subtract* 12, we may say in Algebra *to add* -12. The Algebraical statement

$$-2-7-10=-19$$

may therefore be read, in Arithmetical language, *the subtraction of* 2, 7, *and* 10 *from a number is equivalent to the subtraction of* 19; or, in Algebraical language, *the addition of* -2, -7, *and* -10 *to a number is equivalent to the addition of* -19.

As an illustration of the foregoing phraseology we may take the following example: Suppose a man's gains to be denoted by $+$, and his losses by $-$; then the statement

$$-200-60-500=-760$$

may be read, if a dollar is the unit understood, *the sum of a loss of* 200 *dollars, a loss of* 60 *dollars, and a loss of* 500 *dollars is equivalent to a loss of* 760 *dollars.* It may also be read, *the subtraction of a gain of* 200 *dollars, a gain of* 60 *dollars, and a gain of* 500 *dollars is equivalent to the subtraction of a gain of* 760 *dollars.*

22. (iii.) POSITIVE AND NEGATIVE QUANTITIES.

Since $+5$ denotes a number 5 *to be added*, and -2 a number 2 *to be subtracted*, the performance of both these operations may be denoted by $+5-2$; and since the performance of these two operations is equivalent to adding 3, we may write

$$+5-2=+3.$$

In Algebra $+5-2$, or $+3$, is called the sum of $+5$ and -2.

Thus, $+7-5$, or $+2$, is the sum of $+7$ and -5; $+\frac{3}{4}-\frac{1}{2}$, or $+\frac{1}{4}$, is the sum of $+\frac{3}{4}$ and $-\frac{1}{2}$.

Again, since to add 2 to a number and then subtract 5 is equivalent to subtracting 3, we may write

$$+2-5=-3;$$

that is, in Algebraical language, *to add* $+2$ *and* -5 *to a number is equivalent to adding* -3.

The statement

$$+7-5+2=+4$$

may therefore be read, in Arithmetical language, *to add* 7 *to, then subtract* 5 *from, and finally add* 2 *to a number is equivalent to adding* 4; or, in Algebraical language, *the sum of* $+7$, -5, *and* $+2$ *is* $+4$.

So also the statement

$$-3+10-15=-8$$

may be read, in Arithmetical language, *to subtract* 3 *from, then add* 10 *to, and finally subtract* 15 *from a number is equivalent to subtracting* 8; or, in Algebraical language, *the sum of* -3, $+10$, *and* -15 *is equal to* -8.

As an illustration of the foregoing phraseology, we may again take the case of a man's gains and losses. Thus the statement

$$+10-8=+2$$

may be read, if a dollar is the unit understood, *a gain of* 10 *dollars and a loss of* 8 *dollars are equivalent to a gain of* 2 *dollars.* So also the statement

$$-25+20=-5$$

may be read *a loss of* 25 *dollars and a gain of* 20 *dollars are equivalent to a loss of* 5 *dollars.*

23. From the preceding cases we can deduce the following rule for finding the sum of any positive and negative numbers.

(i.) When the signs are all alike—*Find their Arithmetical sum, and prefix the common sign.*

(ii.) When the signs are different—*Find the numerical difference between the Arithmetical sum of the positives and the Arithmetical sum of the negatives, and prefix the sign of the numerically greater sum.*

Examples.

(1.) The sum of $+4$, $+3$, $+\frac{1}{2}$, and $+7$ is

$$+4+3+\tfrac{1}{2}+7=+14\tfrac{1}{2}.$$

(2.) The sum of -5, -12, $-\frac{3}{4}$, and -3 is

$$-5-12-\tfrac{3}{4}-3=-20\tfrac{3}{4}.$$

(3.) The sum of $+4$, -2, -3, $+5$, and $+7$ is

$$\begin{aligned} +4-2-3+5+7 &= +4+5+7-2-3 \\ &= +16-5 \\ &= +11. \end{aligned}$$

Here $+16$ is the sum of the positives, and -5 of the negatives; 11 is the numerical difference between these sums, and has the sign of the numerically greater $+16$.

(4.) The sum of $+\frac{1}{4}$, -2, $-\frac{1}{2}$, $+1$, and $-\frac{3}{4}$ is

$$\begin{aligned} +\tfrac{1}{4}-2-\tfrac{1}{2}+1-\tfrac{3}{4} &= +\tfrac{1}{4}+1-2-\tfrac{1}{2}-\tfrac{3}{4} \\ &= +\tfrac{5}{4}-\tfrac{13}{4} \\ &= -2. \end{aligned}$$

Here $+\frac{5}{4}$ is the sum of the positives, and $-\frac{13}{4}$ of the negatives; 2 is the numerical difference between these sums, and has the sign of the numerically greater $-\frac{13}{4}$.

Exercise V.

Find the sum of

(1.) $+2, +5, +18$. (2.) $+\frac{1}{3}, +3, +\frac{3}{4}$.

(3.) $-8, -13, -7$. (4.) $-\frac{3}{4}, -1, -\frac{4}{5}$.

(5.) $+18, -13$. (6.) $-26, +20$.

(7.) $+\frac{3}{4}, -\frac{2}{3}$. (8.) $-\frac{8}{9}, +\frac{3}{4}$. (9.) $+2{\cdot}5, -3{\cdot}2$.

(10.) $+2, -3, +12$. (11.) $-3, +4, -6, +7$.

(12.) $+5, -8, -12, +3$. (13.) $+\frac{1}{2}, -1, +\frac{3}{4}, -\frac{6}{7}$.

(14.) $-\frac{2}{3}, -\frac{1}{5}, +2, -\frac{1}{6}, +\frac{1}{2}$.

(15.) $+2{\cdot}58, -3{\cdot}26, +1{\cdot}089, -0{\cdot}067$.

II. Letter Symbols.

24. (i.) Like Quantities.

We shall now show how to find the sum of quantities whose numerical values are represented by letters. When these quantities are like quantities, their sum is obtained by the rule—

The sum of any number of like quantities is a like quantity whose coefficient is the sum of the several coefficients.

Examples.

(1.) The sum of $+2a$, $+5a$, and $+10a$ is

$$+2a+5a+10a=+17a.$$

Here $+17$ is the sum of $+2$, $+5$, and $+10$.

(2.) The sum of $-3c$, $-10c$, and $-12c$ is

$$-3c-10c-12c=-25c.$$

Here -25 is the sum of -3, -10, and -12.

(3.) The sum of $+8x$, $-12x$, and $+7x$ is

$$+8x-12x+7x=+3x.$$

Here $+3$ is the sum of $+8$, -12, and $+7$.

(4.) The sum of $-10x$, $+12x$, and $-5x$ is

$$-10x+12x-5x=-3x.$$

Here -3 is the sum of -10, $+12$, and -5.

Exercise VI.

Find the sum of

(1.) $+a,\ +2a.$ (2.) $+3a,\ +5a,\ +7a.$ (3.) $-a,\ -4a$
(4.) $-2a,\ -6a,\ -5a.$ (5.) $+5x^2,\ -3x^2.$
(6.) $-a^2,\ +4a^2.$ (7.) $+4a,\ -7a.$ (8.) $-2c,\ +5c,\ +7c.$
(9.) $-10c,\ +8c,\ -3c.$ (10.) $+x^2,\ -7x^2,\ +3x^2,\ -x^2.$
(11.) $-2ab,\ +11ab,\ +ab,\ -3ab.$ (12.) $-\frac{1}{2}a,\ +\frac{2}{3}a.$
(13.) $+\frac{4}{5}a^2,\ -\frac{6}{7}a^2.$ (14.) $-2a,\ +\frac{1}{3}a,\ -a.$
(15.) $+a,\ -\frac{4}{5}a,\ +\frac{1}{3}a,\ -2a.$

25. (ii.) Unlike Quantities.

The sum of any number of unlike quantities is denoted by writing them in a row in any order, with their proper signs between them; and each quantity is called a *term* of the sum.

Thus the sum of $+2a$ and $+3b$ is $+2a+3b$; of $+3x$ and $-5y$ is $+3x-5y$; of $-\frac{1}{2}a^2$ and $-2b^2$ is $-\frac{1}{2}a^2-2b^2$; of $+2a$, $+3b$, and $-5c$ is $+2a+3b-5c$. The terms of $-2x+5y$ are $-2x$ and $+5y$; and of $-6a+7b-3c$ are $-6a$, $+7b$, and $-3c$.

26. If a quantity contains no parts connected by the sign $+$ or $-$ it is called a *mononomial.*

Thus $+2x$, $-3ab$, $+6x^2y$, are mononomials.

27. When a quantity consists of two terms it is called a *binomial* expression; when it consists of three terms it is called a *trinomial* expression; and generally when it consists of several terms it is called a *polynomial,* or *multinomial* expression.

Thus $+2a-3b$ is a binomial, and $+a-2b+3c$ a trinomial expression.

28. The sign of a mononomial, or of the first term of a polynomial, if it is positive, is generally omitted.

Thus $2x^2$ stands for $+2x^2$, and $a-b+c$ for $+a-b+c$.

29. Like terms when they occur must be added together. The operation may be conducted by arranging the several

quantities in rows under each other, so that like terms shall stand in the same column.

Examples.

(1.) Find the sum of $2a+3b$ and $5a-2b$.

$$\begin{array}{l} 2a+3b \\ 5a-2b \\ \hline 7a+b. \end{array}$$

Here $7a$ is the sum of $2a$ and $5a$, and $+b$ of $+3b$ and $-2b$.

(2.) Find the sum of $xy-6x$ and $x-8xy$.

$$\begin{array}{r} -6x+xy \\ x-8xy \\ \hline -5x-7xy. \end{array}$$

Here $-5x$ is the sum of $-6x$ and x, and $-7xy$ of $+xy$ and $-8xy$.

(3.) Find the sum of $3+x+xy-8x^2$, $-3xy+2-6x$, and $4xy+x^2+1$.

$$\begin{array}{llll} 3+ & x\ + & xy & -8x^2 \\ 2- & 6x- & 3xy & \\ 1 & & +4xy & +x^2 \\ \hline 6- & 5x+ & 2xy & -7x^2. \end{array}$$

Here the sum of 1, 2, 3 in the first column is 6; of $+x$, $-6x$ in the second is $-5x$; of $+xy$, $-3xy$, $+4xy$ in the third is $+2xy$; and of $-8x^2$, $+x^2$ in the fourth is $-7x^2$.

(4.) Add together $\frac{1}{2}a-\frac{1}{3}b-\frac{1}{4}c$, $\frac{1}{2}b+\frac{1}{3}c+\frac{1}{4}a$, $\frac{1}{2}c-\frac{1}{3}a-\frac{1}{4}b$.

$$\begin{array}{l} \frac{1}{2}a-\frac{1}{3}b-\frac{1}{4}c \\ \frac{1}{4}a+\frac{1}{2}b+\frac{1}{3}c \\ -\frac{1}{3}a-\frac{1}{4}b+\frac{1}{2}c \\ \hline \frac{5}{12}a-\frac{1}{12}b+\frac{7}{12}c. \end{array}$$

Here $\frac{5}{12}a$ is the sum of $\frac{1}{2}a$, $\frac{1}{4}a$, and $-\frac{1}{3}a$; $-\frac{1}{12}b$ and $+\frac{7}{12}c$ are the sums of the quantities in the second and third columns, respectively.

Exercise VII.

Find the sum of

(1.) $2a, -3b$. (2.) $-x, +3y$. (3.) $-2x, -3y, -z$.

(4.) $3a, 2x, -5y$. (5.) $4, -x, 2y$. (6.) $a^2, -b^2, \frac{1}{2}$.

(7.) $a, -2b, 3c, -d$. (8.) $-x, 2y, -z, 1$.

(9.) $a-3b, 3a+b$. (10.) $-2a^2+bc, 3a^2-5bc$.

(11.) $3a-5b, -4a+2b, 5a-6b$.

(12.) $a-3b, 2b-5c, 4c-3a$.

(13.) $4x-3y+2z, -3x+y-4z, x-4y+z$.

(14.) $a-x+3, 5a+2x-5, -2a+7$.

(15.) $2x-7, 5a+4, -6x+3$.

(16.) $a-2b+3c, b-2c+3a, c-2a+3b$.

(17.) $x-2y+3z-1, 2x+3-4z, 5y-2+7x$.

(18.) $x^2+2ax+a^2, 2x^2-2a^2, x^2-2ax+a^2$.

(19.) $3a^3+a^2b-2ab^2+b^3, 3ab^2-2a^2b+a^3, a^2b-ab^2+3b^3$.

(20.) $a-b+\frac{c}{6}, a+b-\frac{c}{3}, b-a+\frac{c}{6}$.

(21.) $\frac{x}{2}-\frac{y}{3}+\frac{z}{4}, \frac{y}{2}-\frac{z}{3}+\frac{x}{4}, \frac{z}{2}-\frac{x}{2}+\frac{y}{4}$.

(22.) $a+5b-c, 2a-4b-c, \frac{b}{2}-a+\frac{c}{3}$.

CHAPTER III.

SUBTRACTION.

30. The *Algebraical difference* between one quantity and another is the quantity which added Algebraically to the latter will produce the former.

Thus the difference between 2 and -4 is the quantity which added to -4 will produce 2; the difference between $-5a$ and $2a$ is the quantity which added to $2a$ will produce $-5a$; and the difference between $3x$ and $2x-5$ is the quantity which added to $2x-5$ will produce $3x$.

31. The quantity to be diminished is called the *minuend*, and the quantity to be subtracted the *subtrahend*.

32. The difference between two quantities is found by the rule—

Add the first quantity to the second with its sign or signs changed.

The reason for this rule will appear from the following

Examples.

(1.) From 5 take 3.

Here the difference is the sum of 5 and $-3=5-3=2$, because 2 added to 3 makes 5.

(2.) From 7 take -4.

Here the difference is $7+4=11$, because the sum of 11 and $-4=11-4=7$.

(3.) From -6 take -4.

Here the difference $= -6+4= -2$, because the sum of -2 and $-4=-2-4=-6$.

(4.) From -8 take 5.

The difference $=-8-5=-13$, because the sum of -13 and $5=-13+5=-8$.

(5.) From $2a$ take $-3a$.

The difference $=2a+3a=5a$, because the sum of $5a$ and $-3a=5a-3a=2a$.

(6.) From $-5x$ take 4.

The difference $=-5x-4$, because the sum of $-5x-4$ and 4 is $-5x$.

(7.) From $2a$ take $-3a+2b$.

The difference = the sum of

$$2a \text{ and } 3a-2b=2a+3a-2b=5a-2b,$$

because the sum of $5a-2b$ and $-3a+2b$ is $2a$.

The operation of changing signs and adding may be performed mentally, and the difference exhibited as in the following examples, in which the minuend and subtrahend are written in rows with the difference underneath. In this arrangement the first row is equal to the sum of the second and third rows.

(8.) From $3x^2+y$ take x^2-5y.

$$\begin{array}{r} 3x^2+y \\ x^2-5y \\ \hline 2x^2+6y \end{array}$$

Here $2x^2$ is the sum of $3x^2$ and $-x^2$; and $+6y$ of $+y$ and $+5y$.

(9.) From $5a+3b-c$ take $a-b+3c$.

$$\begin{array}{r} 5a+3b-c \\ a-b+3c \\ \hline 4a+4b-4c \end{array}$$

Here $4a$ is the sum of $5a$ and $-a$; $+4b$ of $+3b$ and $+b$; and $-4c$ of $-c$ and $-3c$.

33. From the foregoing examples it appears that, in Algebraical language, *to subtract a positive quantity* is equivalent to *adding a negative;* and *to subtract a negative quantity* is equivalent to *adding a positive.*

This phraseology may be illustrated by taking the case of a man's gains and losses to be denoted by + and −, respectively. Thus, *to subtract a gain of* 10 *dollars* is equivalent to *adding a loss of* 10 *dollars;* and *to subtract a loss of* 25 *dollars* is equivalent to *adding a gain of* 25 *dollars.*

Moreover, if a man gains a dollars and loses b dollars, we say, in Arithmetical language, either that he gains $a-b$ dollars, if a is greater than b, or loses $b-a$ dollars, if b is greater than a. Either of these phrases may be employed indifferently if we agree that *a gain of* $-c$ *dollars* means *a loss of* c *dollars,* and that *a loss of* $-c$ *dollars* means *a gain of* c *dollars.*

Thus if a man gains 10 dollars and loses 5 dollars, we may either say that he gained $10-5$, or 5 dollars, or that he lost $5-10$, or -5 dollars. Again, if he gains 8 dollars and loses 12 dollars, we may either say that he gained $8-12$, or -4 dollars, or that he lost $12-8$, or 4 dollars.

Exercise VIII.

(1.) From 1 take -3.

(2.) From 1 take -1.

(3.) From -5 take 4.

(4.) From 12 take 15.

(5.) From -3 take -8.

(6.) From -8 take -5.

(7.) From $4{\cdot}56$ take $-6{\cdot}04$.

(8.) From $-1{\cdot}04$ take $2{\cdot}35$.

(9.) From $-4{\cdot}32$ take $-2{\cdot}16$.

(10.) From $-1{\cdot}089$ take $-0{\cdot}123$.

(11.) From $2a$ take $3a$.

(12.) From $-5x$ take $2x$.

(13.) From $8a^2$ take $-2a^2$.

(14.) From $-3c$ take $-5c$.

(15.) From $2a$ take $\frac{b}{2}$.

(16.) From $-a^2$ take $2a^2$.

(17.) From $5x^2$ take $-6x^2$.

(18.) From $7a$ take $4a-b$.

(19.) From $a+x$ take $a-x$.

(20.) From $5a-2x+3$ take $2a-x-1$.

(21.) From $3a^2-4ab+b^2$ take $3a^2+ab-b^2$.

(22.) From $ax-4by+3cz$ take $2ax+by-cz$.

(23.) From $-4a+b-1$ take $2a-x+3$.

(24.) From $12x^2-5x+1$ take $7x^3-16x^2+1$.

(25.) From $\frac{3x}{4}+y+\frac{z}{2}$ take $\frac{x}{2}+\frac{y}{2}-z$.

(26.) From $\frac{3}{2}a+b-\frac{1}{3}c$ take $a+\frac{1}{2}b-\frac{1}{6}c$.

CHAPTER IV.

THE USE OF DOUBLE SIGNS AND OF BRACKETS.

34. THE operations of addition and subtraction of positive and negative quantities may also be denoted by the use of $+$ for the former operation and $-$ for the latter.

Thus instead of saying *add together* $2a$ *and* $-3b$, we may employ the notation $2a+\ -3b$, the equivalent of which is of course $2a-3b$. So also *the sum of* $-5a^2$, $+3b$, *and* $-2c$ may be written $-5a^2+\ +3b+\ -2c$, which is equivalent to $-5a^2+3b-2c$; and *the difference between* $5x$ *and* $-7y$ may be expressed $5x\ -\ -7y$, the equivalent of which is $5x+7y$.

According to this notation, therefore, $2a^2+\ -3b+\ +2c$ means *the sum of* $2a^2$, $-3b$, and $+2c$; $-5x+\ +8y+\ -1$ *the sum of* $-5x$, $+8y$, and -1; $7a-\ +2b$ *the difference between* $7a$ and $+2b$; $-2x-\ -5$ *the difference between* $-2x$ and -5; $4a^2-\ -7$ *the difference between* $4a^2$ and -7.

35. When any of the quantities before which the double signs are to be used contains more terms than one, it must be enclosed in a bracket; thus $+\ (2a-3b)$, $+\ (-x+4)$, $-\ (x-5)$, $-\ (-2+a^2+x)$.

Thus $a+(3b-c)$ denotes the sum of a and $3b-c$; $2x+(4y-2)+(-2z+1)$ the sum of $2x$, $4y-2$, and $-2z+1$; $2a^2-b-(3a^2+4)$ the difference between $2a^2-b$ and $3a^2+4$; and $2a+(b-1)-(c+4)$ the sum of $2a$ and $b-1$ less $c+4$.

EXERCISE IX.

Retaining the given quantities, denote by using the double signs and brackets (when necessary) in the following operations:—

(1.) The sum of $2x^2$, -1; $3x$, $-4y$, -5; $2a$, $-3b$, $+4c$.
(2.) From $2a$ take $-5a$.
(3.) From -6 take $+5x$.
(4.) From the sum of $2a$ and $-3b$ take $+7$.
(5.) From the sum of 5 and $+x$ take $-3a$.
(6.) The sum of $5a$ and $b-4$.
(7.) The sum of $-a$ and $-b+5$.
(8.) The sum of $a-4$ and $2b-c$.
(9.) The sum of x^2 and $2y+5$ less z.
(10.) The sum of $a-1$ and $3b+5$ less $-3c$.
(11.) The sum of x, $2x^2-1$, and $-3x^2-8$.
(12.) The difference between $4a^2$ and b^2-c.
(13.) The difference between a^2+4 and $-2b+3$.
(14.) The difference between $2a-5$ and a^2-2a+3.
(15.) From the sum of $a+b+c$ and $a-b-c$ take $-a+2b-3c$.

36. Double signs may be equivalently replaced by single ones by the rule:—

Like signs produce $+$, *and unlike signs* $-$; that is

$$++=--=+,$$
$$+-=-+=-.$$

Thus $a++5=a+5$; $2x--a=2x+a$; $3+-4c=3-4c$; $c-+2a=c-2a$.

37. Expressions may be cleared of brackets by the rule:—

The sign $+$ *before a bracket does not change the signs within, whilst the sign* $-$ *changes every sign within.*

Thus
$$4+(b-c)=4+b-c,$$
$$2a+(-x+c-2d)=2a-x+c-2d,$$
$$4a^2-1-(2b^2+c)=4a^2-1-2b^2-c,$$
$$3x-(-4y+5)=3x+4y-5,$$
$$x-(y-z)+4-(-3y+x)=x-y+z+4+3y-x.$$

EXERCISE X.

Replace the double signs by single ones in the expressions:—

(1.) $2a++3b+-c.$ (2.) $ab-+bc--c.$

(3.) $x^2+-4x+-1.$ (4.) $5x^3+-3x^2--7x-+8.$

Clear of brackets:—

(5.) $8a-(b+c).$ (6.) $8a-(b-c).$

(7.) $8a-(-2b+3c).$ (8.) $2a-1+(b-5)+c.$

(9.) $x+5-(2-4y)+8.$ (10.) $a+(b-c)-(a-c).$

(11.) $3x^2-1-(-x+4)+2x-(x^2-5).$

CHAPTER V.

MULTIPLICATION.

38. When it is desired to denote the operation of multiplying several expressions together so as to exhibit the various factors, we enclose each in a bracket and write them together in a row in any order.

Thus $(+2a)(-3b)$ denotes the product of $+2a$ and $-3b$; $(2a-1)(-b)$ the product of $2a-1$ and $-b$; $(x^2-3)(2x+5)$ the product of x^2-3 and $2x+5$; and $(x-1)(x+2)(2x-5)$ the product of $x-1$, $x+2$, and $2x-5$.

39. Each of the quantities so enclosed in brackets is called a *factor* of the product.

Thus $-2a$, a^2-1, and $2a-3$ are the factors of $(-2a)(a^2-1)$ $(2a-3)$.

40. When a factor is mononomial, it is called a *simple* factor; otherwise a *compound* factor.

Thus in the preceding example $-2a$ is a simple factor, and a^2-1 and $2a-3$ compound factors.

41. In the case of a simple factor the bracket may be omitted (i.) if the simple factor is written in the first place; (ii.) if the sign of the simple factor is not expressed.

Thus we generally write $-2a(x-1)$, $x(x^2-2)$, $(a-b)x$, $(a-b)(c-d)x$.

If the sign of a simple factor is expressed, the bracket may be replaced by a multiplication sign, $\times$.

Thus $\quad -2a \times +3b = -2a(+3b),$
$3x \times -5y = 3x(-5y),$
$(a-1) \times -2a^2 = (a-1)(-2a^2).$

Exercise XI.

Express in Algebraical language, retaining the given factors and using brackets when necessary:—

(1) The products of $a-1, 2a^2-3$; $-2+a, -3-a^2$; $x-5, -2x+7$.

(2.) The products of $-2a^2, b^2-1$; $a^2-1, -3a$; $5x, -x^2+3$.

(3.) The products of $\frac{1}{3}, x-1$; $\frac{1}{5}, 2x-3$; $-\frac{2}{3}, x^2-5$.

(4.) The products of $-5x, x-1, x+2$; $x^2-4, +5x, 2a+3$.

(5.) The products of $+8x, -5y, xy-1$; $-7a, ab-3, +8b$.

42. The mode of performing the operation of multiplication whereby products are expressed as mononomials or polynomials will now be explained. It is convenient to make three cases:

I. The Multiplication of Simple Factors.
II. The Multiplication of a Simple and a Compound Factor.
III. The Multiplication of Compound Factors.

43. I. The product of two simple factors is obtained by the following rule:—

(i.) *The sign of the product is obtained by the rule of signs: like signs produce +, and unlike signs —.*

Thus the signs of $+2a\ (+3b)$, $-2a\ (-3b)$, $+2a\ (-3b)$, $-2a\ (+3b)$ are, respectively, +, +, —, —.

(ii.) *The numerical coefficient of the product is the product of the numerical coefficients of the factors.*

Thus the numerical value of the coefficient of the product of $-2a$ and $+3bc$ is $2 \times 3 = 6$.

(iii.) *The literal part of the product is the product of the literal parts of the factors* (9).

Thus the literal part of the product of $-3x$ and $4y^2z$ is xy^2z.

(iv.) *The product of two powers of the same letter is a power whose index is the sum of the indices of the factors.*

Thus
$$a \cdot a^2=a^{1+2}=a^3,$$
$$a^2 \cdot a^3=a^{2+3}=a^5,$$
$$a^4 \cdot a^7=a^{4+7}=a^{11}.$$

Examples.

(1.) The product of $+2a$ and $-3b=+-2\times 3ab=-6ab$.

(2.) The product of $-5a$ and $2d=-5\times 2ad=-10ad$.

(3.) The product of $-\frac{1}{2}a$ and $+3b^2=-+\frac{1}{2}\times 3ab^2=-\frac{3}{2}ab^2$.

(4.) $-ab(-3c)=--3abc=+3abc$.

(5.) $-\frac{1}{2}a^2(+\frac{1}{3}b)=-+\frac{1}{2}\cdot\frac{1}{3}a^2b=-\frac{1}{6}a^2b$.

(6.) $-2x(+3x^2)=-+2\times 3x\cdot x^2=-6x^3$.

(7.) $-6a^3(-3a^4)=--6\times 3\ a^3\cdot a^4=+18a^7$.

(8.) $+2a^2bc^3(-5ab^2c^4)=+-2\times 5\ a\cdot a^2\cdot b\cdot b^2\cdot c^3\cdot c^4=$ $-10a^3b^3c^7$.

EXERCISE XII.

Find the product of

(1.) $+3a,\ -2b;\ -a,\ +5c;\ -2a^2,\ -3b;\ 5x,\ -6y$.

(2.) $2ab,\ -7c^2;\ -4a^2,\ +5bc;\ -2x,\ 8yz;\ -6,\ -8a$.

(3.) $-\frac{1}{2}x,\ 3y;\ \frac{3}{4}a,\ -2b;\ -\frac{1}{2}x,\ -\frac{1}{3}y;\ 2a^2,\ -\frac{b}{5}$.

(4.) $2xy^2,\ -3x^2y;\ -ax^2y,\ -3xy^4;\ \frac{4}{5}ab^2c,\ -\frac{3}{4}a^2bc^3$.

(5.) $\frac{a}{2},\ -\frac{2a^2}{3};\ \frac{ab}{5},\ \frac{a^2b^3}{3};\ -\frac{2xy^2}{5},\ \frac{3x^2y^3}{8};\ -\frac{axy^2}{4},-\frac{bx^2}{3}$.

44. II. *The product of a simple and a compound factor is the Algebraical sum of the products of the simple factor and the several terms of the compound factor.*

Thus the product of $2a$ and $3b-5c+d$ is the sum of the products of $2a$, $3b$; $2a$, $-5c$; and $2a$, $+d$; and is therefore equal to $6ab-10ac+2ad$.

The work may be arranged as in the following

Examples.

(1.) Find the product of $-3x$ and $2y^2-4xy-5$.

$$\begin{array}{r} 2y^2-4xy-5 \\ -3x \\ \hline -6xy^2+12x^2y+15x. \end{array}$$

(2.) Find the product of $2x^2-\frac{1}{3}x+\frac{1}{5}$ and $-\frac{2}{7}xy$.

$$\begin{array}{r} 2x^2-\frac{1}{3}x+\frac{1}{5} \\ -\frac{2}{7}xy \\ \hline -\frac{4}{7}x^3y+\frac{2}{21}x^2y-\frac{2}{35}xy. \end{array}$$

(3.) Clear of brackets the expression $-2a\,(3a^2-5a+1)$. As this expression denotes the product of $-2a$ and $3a^2-5a+1$, it is equivalent to $-6a^3+10a^2-2a$.

Exercise XIII.

Find the product of

(1.) $4a-3b+c$, $-2x$; $3x^2-2x+1$, $4y$; $2ab-3c$, $-d$.
(2.) x^2-2x-5, $3x$; $2a^2-3a+7$, $-a^3$; $x^2-ax+2a^2$, $-4ax$.
(3.) $2x+y-3z$, $2xyz$; $7a^2b-ab^2+ab$, $-4ab$.
(4.) $a+\frac{1}{2}b-\frac{3}{5}c$, $2ab$; $2a^2-3a+4$, $-\frac{1}{2}a$; $\frac{2}{3}x^2-ax+\frac{3}{4}a^2$, $\frac{1}{2}ax$.
(5.) $\frac{4a}{5}-\frac{2ab}{3}+1$, -15; $\frac{2x}{7}-1+\frac{3ax}{2}$, $\frac{14}{15}ax$.

Clear of brackets:—

(6.) $5(x^2-x+4)$; $-2(a-ab+3)$; $-a(a^2-2ax+1)$.
(7.) $(a-x)x$; $(a-b+c)c$; $(-1+ab-3a^2)5ab$.
(8.) $\frac{2a}{3}(6a^2-9a+12)$; $-\frac{x^2}{5}(-10+2x-15x^3)$.

45. III. *The product of two compound factors is the Algebraical sum of the products of one factor and the several terms of the other.*

Thus the product of $2a-3b$ and $4c+5d$ is the sum of the products $2a-3b$, $4c$ and $2a-3b$, $+5d$, and is, therefore, equal to $8ac-12bc+10ad-15bd$.

The work may be conveniently arranged as in the following

Examples.

(1.) Multiply $2x^2-3x+5$ by $4x-7$.

$$\begin{array}{l} 2x^2-3x+5 \\ \underline{4x\ -7\qquad\quad} \\ 8x^3-12x^2+20x \\ \underline{\qquad\ -14x^2+21x-35} \\ 8x^3-26x^2+41x-35. \end{array}$$

Here $8x^3-12x^2+20x$ is the product of $4x$ and $2x^2-3x+5$; $-14x^2+21x-35$ is the product of -7 and $2x^2-3x+5$; and the sum of these two partial products, which are arranged so that like terms stand in the same column, is the product required.

It will be observed that in the foregoing process we work from left to right, and not from right to left, as in the corresponding Arithmetical operation.

(2.) Multiply $2a-b$ by $c-3d$.

$$\begin{array}{l} 2a-b \\ \underline{\ c-3d} \\ 2ac-bc \\ \underline{\qquad\qquad -6ad+3bd} \\ 2ac-bc-6ad+3bd. \end{array}$$

In this case, as there are no like terms in the partial products, the second is placed entirely to the right of the first.

(3.) Multiply $1-2x+3x^2$ by $4x-5x^2+2$.

It will be found most convenient in this and similar examples to arrange the given factors according to ascending or descending powers of x; that is, so that the exponents of the successive terms shall continually increase or decrease. In the former arrangement the numeral stands first, in the latter last.

In the present case let them be arranged in the order $1-2x+3x^2$, $2+4x-5x^2$.

$$
\begin{array}{lllll}
1 & -2x & +3x^2 & & \\
2 & +4x & -5x^2 & & \\
\hline
2 & -4x & +6x^2 & & \\
 & +4x & -8x^2 & +12x^3 & \\
 & & -5x^2 & +10x^3 & -15x^4 \\
\hline
2 & & -7x^2 & +22x^3 & -15x^4.
\end{array}
$$

(4.) Multiply $2a^2-ab+b^2$ by $a^2+ab-3b^2$.

$$
\begin{array}{lllll}
2a^4 & & & & \\
\end{array}
$$

$$
\begin{array}{lllll}
2a^2 & -ab & +b^2 & & \\
 & a^2 & +ab & -3b^2 & \\
\hline
2a^4 & -a^3b & +a^2b^2 & & \\
 & +2a^3b & -a^2b^2 & +ab^3 & \\
 & & -6a^2b^2 & +3ab^3 & -3b^4 \\
\hline
2a^4 & +a^3b & -6a^2b^2 & +4ab^3 & -3b^4.
\end{array}
$$

Here the factors are arranged according to descending powers of a and ascending powers of b.

EXERCISE XIV.

Multiply together

(1.) $2x-3$, $x+4$; $4x+5$, $-x+1$; $2-3x$, $1+x$.

(2.) x^2-2, $2x-1$; $1-x$, $2+3x^2$; $1+x^2$, $3-x^3$.

(3.) $2a^2-a+4$, $3a-2$; $1-a+a^2$, $1+a$; $1+a+a^2$, $1-a$.

(4.) $a+b$, $m-n$; $a+b-c$, $m+2n$; $2m-n$, $2m+n$.

(5.) $xy+y^2$, $xy-x^2$; $x+xy-y^2$, x^2-2y.

(6.) $2a^2-5a+1$, a^2+3a-4; $a+2b-3c$, $a-2b+3c$.

(7.) $3a+2b-c$, $a-2b+3c$; $3x^2+2xy+y^2$, $x^2-2xy+3y^2$.

(8.) $x^2-\frac{1}{3}$, $x^2-\frac{1}{2}$; $a+\frac{1}{2}$, $a-\frac{1}{3}$; $2a-\frac{1}{2}$, $a+\frac{1}{4}$.

(9.) $x^2-\frac{1}{2}x+1$, $2x-\frac{1}{5}$; $3x^2-\frac{2}{3}x+\frac{1}{6}$, $3x-\frac{1}{2}$.

(10.) $x+\frac{y}{2}-2$, $\frac{x}{4}-3y$; $\frac{x^2}{3}-2x+\frac{1}{4}$, $3x-\frac{2}{3}$.

(11.) $\frac{a^2}{2}-\frac{2a}{3}+1$, $\frac{a^2}{2}+\frac{2a}{3}-1$.

(12.) $x^2+y^2-xy+x+y-1$, $x+y-1$.

(13.) $a^2+b^2+c^2-bc-ca-ab$, $a+b+c$.

46. *The product of three expressions is found by multiplying the product of two of them by the third.*

Examples.

(1.) Multiply together $2x$, $-3x^2y$, $-\frac{3}{4}xy^2z$.
Here $2x(-3x^2y)=-6x^3y$; and $-6x^3y(-\frac{3}{4}xy^2z)=+\frac{9}{2}x^4y^3z$.

(2.) Multiply together $x-1$, $x-2$, $x-3$.

$$\begin{array}{l} x-1 \\ x-2 \\ \hline x^2-x \\ \quad -2x+2 \\ \hline x^2-3x+2 \\ x-3 \\ \hline x^3-3x^2+2x \\ \quad -3x^2+9x-6 \\ \hline x^3-6x^2+11x-6. \end{array}$$

EXERCISE XV.

Multiply together

(1.) $-3a^2$, $+2a^2b$, $-5ab^2$; $\frac{1}{2}x$, $-\frac{1}{3}x^2$, $+\frac{2}{5}x^4$; $-8x^2y$, $-\frac{2}{3}y^2z$, xyz.

(2.) $-2x$, $3xy$, $4x^2-5y$; $2ab$, $-\frac{1}{2}a^2$, $2a^2-3ab+1$.

(3.) $2x-3$, $4x+1$, $x-2$.

(4.) x^2-x+1, $x-1$, $x+1$.

(5.) $x^2+2ax+a^2$, $x^2-2ax+a^2$, $x^4+2a^2x^2+a^4$.

CHAPTER VI.

DIVISION.

47. Division being the inverse of multiplication, it follows that, when two factors are multiplied together, either factor will be the *quotient* of the product divided by the other.

Thus, since $+a(-2b)=-2ab$, $+a$ is the quotient of $-2ab$ divided by $-2b$, and $-2b$ is the quotient of $-2ab$ divided by $+a$.

Again, since $-2x^2(x^3-4x+3)=-2x^5+8x^3-6x^2$, it follows that x^3-4x+3 is the quotient of $-2x^5+8x^3-6x^2$ divided by $-2x^2$, and $-2x^2$ is the quotient of $-2x^5+8x^3-6x^2$ divided by x^3-4x+3.

48. When it is desired to denote that one quantity is to be divided by another, they are enclosed in brackets and written in a row with the sign $\div$ between them; or the second quantity is written below the first with a line between them.

Thus $(-2x^2)\div(+3x)$, or $\frac{-2x^2}{+3x}$, denotes that $-2x^2$ is to be divided by $+3x$; $(3x^2-2x+5)\div(x-4)$, or $\frac{3x^2-2x+5}{x-4}$, that $3x^2-2x+5$ is to be divided by $x-4$.

49. The first or upper quantity is called the *dividend*, and the second or lower the *divisor*.

50. The bracket is generally omitted in the case of a mononomial.

Thus $(-2x)\div(-3y)$ is written $-2x\div-3y$; $(2x^2-1)\div(4x)$ is written $(2x^2-1)\div4x$; $(-3x^3)\div(2x-1)$ is written $-3x^3\div(2x-1)$.

EXERCISE XVI.

Retaining the given quantities and employing brackets only when necessary, express in Algebraical language—

(1.) Divide $2a-5$ by $-3a$; $4a^2-3a+1$ by $3a-4$.

(2.) Divide $2a$ by $-3b$; $-x^2$ by $+2x$; $3x$ by $2a$.

(3.) Divide $4x^2$ by $2x-5$; $-ax^2$ by $x-a$.

51. The mode of performing the operation of division whereby quotients are expressed as mononomials or polynomials will now be explained. We shall consider in order three cases:

I. WHEN THE DIVIDEND AND DIVISOR ARE MONONOMIALS.
II. WHEN THE DIVISOR ONLY IS A MONONOMIAL.
III. WHEN THE DIVIDEND AND DIVISOR ARE POLYNOMIALS.

52. I. The quotient of one mononomial divided by another is obtained by the following rule:

(i.) *The sign of the quotient is obtained by the rule of signs: like signs produce* $+$, *and unlike signs* $-$.

Thus the signs of $+2ab \div 3cd$, $-3x^2 \div -2x$, $4a^2 \div -2a$, $-5x^2y \div +xy$, are, respectively, $+$, $+$, $-$, $-$.

This rule follows from the rule of signs in multiplication; thus, since $+a\,(+b)=+ab$, it follows that

$$\frac{+ab}{+a}=+b.$$

So also from the equivalent forms $-a\,(+b)=-ab$, $-a\,(-b)=+ab$, $+a\,(-b)=-ab$, we deduce

$$\frac{-ab}{-a}=+b,\ \frac{+ab}{-a}=-b,\ \frac{-ab}{+a}=-b.$$

(ii.) *The numerical coefficient, without regard to sign, of the quotient is obtained by dividing the numerical coefficient of the dividend by the numerical coefficient of the divisor.*

Thus the numerical coefficient of $12a^2 \div 3a$ is $12 \div 3=4$, of $2x^3 \div 3x$ is $\frac{2}{3}$, and of $\frac{1}{2}a^2 \div \frac{2}{3}a$ is $\frac{1}{2} \div \frac{2}{3}=\frac{3}{4}$.

(iii.) *The literal part of the quotient is obtained by dividing the literal part of the dividend by the literal part of the divisor* (12).

Thus the literal part of the quotient of $2a^2 \div bc$ is $\frac{a^2}{bc}$. In applying this rule the fractional form of expressing the quotient should always be used.

(iv.) *The quotient of two powers of the same letter is a power whose index is the difference of the indices of dividend and divisor.*

The reason for this rule will be evident from the following examples:—

From (10)
$$\frac{a^3}{a}=\frac{aaa}{a}=aa=a^2=a^{3-1};$$
$$\frac{a^5}{a^3}=\frac{aaaaa}{aaa}=aa=a^2=a^{5-3};$$
$$\frac{a^7}{a^4}=\frac{aaaaaaa}{aaaa}=aaa=a^3=a^{7-4};$$

So likewise
$$\frac{a^4}{a^3}=a^{4-3}=a;$$
$$\frac{a^6}{a^2}=a^{6-2}=a^4.$$

53. Since the quotient of any quantity divided by itself is 1, this rule can be applied when the indices of the dividend and divisor are equal, *if the zero power of a letter is considered equal to unity.* Thus $\frac{a^2}{a^2}=1$; and, by the rule, $\frac{a^2}{a^2}=a^{2-2}=a^0$; therefore $a^0=1$. Whenever, therefore, by applying the preceding rule, we get such symbols as x^0, y^0, c^0, we must replace each of them by 1.

Examples.

(1.) $-12 \div +6=-\frac{12}{6}=-2;$
$$-15 \div -10=+\frac{15}{10}=+\frac{3}{2};$$
$$+\frac{1}{2} \div -\frac{3}{4}=-\frac{\frac{1}{2}}{\frac{3}{4}}=-\frac{2}{3}.$$

(2.) $-6a \div 3b=-\frac{6}{3}\cdot\frac{a}{b}=-\frac{2a}{b}.$

(3.) $-2ab \div -5c = +\frac{2}{5}\,\frac{ab}{c} = +\frac{2ab}{5c}$.

(4.) $10a^3 \div -5a^2 = -2a^{3-2} = -2a$.

(5.) $-4a^5 \div -7a^2 = +\frac{4}{7}a^{5-2} = +\frac{4}{7}a^3$.

(6.) $2a^2b^3c^5 \div 3abc^2 = \frac{2}{3}a^{2-1}\,b^{3-1}\,c^{5-2} = \frac{2}{3}ab^2c^3$.

Exercise XVII.

Divide

(1.) -16 by $+4$; 20 by -4; $-\frac{2}{3}$ by $-\frac{5}{6}$; $+5$ by $-\frac{3}{8}$.

(2.) $-a$ by $+2x$; $3a^2$ by $-2b$; $-6xy$ by $-3a$.

(3.) $\frac{x}{2}$ by $-\frac{y}{3}$; $-\frac{2ax}{5}$ by $\frac{b}{3}$; $-\frac{3x^2}{7}$ by $-\frac{2a^2}{3}$.

(4.) $2a^5$ by a^2; $3a^4$ by $3a$; $8x^6$ by $2x^4$.

(5.) $-4a^2b$ by $2ab$; $10a^2b^3c^4$ by $2abc$.

(6.) ax^2y^4 by $-3axy$; $-3a^2xy^2$ by $-5axy$.

54. II. *The quotient of a polynomial divided by a monomial is the Algebraical sum of the quotients of the several terms of the former divided by the latter.*

Thus the quotient of $3x^3-6x^2+8x$ divided by $-2x$ is the sum of the quotients $3x^3 \div -2x$, $-6x^2 \div -2x$, $+8x \div -2x$; and is therefore equal to $-\frac{3}{2}x^2+3x-4$.

The work may be arranged as in the following

Examples.

(1.) Divide $8x^3-4x^2+2x$ by $-2x$.

$$-2x\,)\,\underline{8x^3-4x^2+2x}$$
$$-4x^2+2x-1.$$

(2.) Divide $5a^3b-10a^2b^3+2ab$ by $5ab$.

$$5ab\,)\,\underline{5a^3b-10a^2b^3+2ab}$$
$$a^2-\ 2ab^2\ +\tfrac{2}{5}$$

(3.) Collect coefficients of x in $2ax-3x$.

As this means that $2ax-3x$ is to be expressed as the product of two factors one of which shall be x, we have merely to divide the given quantity by x to get the other factor.

Thus $2ax-3x=(2a-3)x$.

(4.) Collect coefficients of x^2 in $5ax^2-bx^2+x^2$.

By dividing the given quantity by x^2 we get

$$5ax^2-bx^2+x^2=(5a-b+1)\,x^2.$$

Exercise XVIII.

Divide

(1.) $10a-15b+20$ by -5; $-4ax+12-8a^2$ by -4.

(2.) $4a^2x-3a^2+a^3$ by $-a^2$; $12x^3-6x^2+9x$ by $3x$.

(3.) $3a^4-12a^3+15a^2$ by $-3a^2$; $2x^3y-6x^2y^2+8xy^3$ by $2xy$.

(4.) $2a^2bc-3ab^2c+abc^2$ by abc.

(5.) $20a^2bc^2-15ab^2c^3+5ab$ by $-5ab$.

(6.) Collect coefficients of x in $ax-bx$, $2ax-cx+x$.

(7.) Collect coefficients of xy in $4xy-axy$, $3x^2y-xy^2$.

55. Since by (54) $\dfrac{x-1}{3}=\dfrac{x}{3}-\frac{1}{3}$, and by (44) $\frac{1}{3}(x-1)=\dfrac{x}{3}-\frac{1}{3}$, it follows that $\dfrac{x-1}{3}$ and $\frac{1}{3}(x-1)$ are equivalent forms. So likewise

$$\frac{2x-3}{4}=\tfrac{1}{4}(2x-3),$$

$$\frac{3\,(x^2-1)}{5}=\tfrac{3}{5}(x^2-1).$$

56. III. When the dividend and divisor are polynomials, the quotient is obtained by the following rule:—

(i.) *Arrange both dividend and divisor according to ascending or descending powers of some letter.*

(ii.) *The first term of the quotient is found by dividing the leading term of the dividend by the leading term of the divisor. The product of the divisor and the first term of the quotient is subtracted from the dividend, giving the first difference.*

(iii.) *The second term of the quotient is found by dividing the leading term of the first difference by the leading term of the divisor. The product of the divisor and the second term of the quotient is subtracted from the first difference, giving the second difference in the process.*

And so on until the last difference is zero.

Examples.

(1.) Divide $6x^3-5x^2-3x+2$ by $2+3x$.

$$
\begin{array}{r|l}
3x+2) & 6x^3-5x^2-3x+2\ (2x^2-3x+1 \\
& \underline{6x^3+4x^2\qquad\qquad} \\
& \quad -9x^2-3x+2 \\
& \quad \underline{-9x^2-6x\quad\ } \\
& \qquad\quad 3x+2 \\
& \qquad\quad \underline{\underline{3x+2}}
\end{array}
$$

Here the divisor is first arranged, like the dividend, according to descending powers of x. The first term of the quotient $2x^2$ is obtained by dividing $6x^3$, the leading term of the dividend, by $3x$, the leading term of the divisor. The product of the divisor $3x+2$ and $2x^2$, which is $6x^3+4x^2$, is then subtracted from the dividend, giving $-9x^2-3x+2$, the first difference. The second term of the quotient $-3x$ is obtained by dividing $-9x^2$, the leading term of the first difference, by $3x$, the leading term of the divisor. The product of the divisor and $-3x$, which is $-9x^2-6x$, is then subtracted from the first difference, giving $3x+2$, the second difference. The third term of the quotient $+1$ is obtained by dividing $3x$, the leading term of the second difference, by $3x$, the leading term of the divisor. The product of the divisor and $+1$, which is $3x+2$, is then subtracted from the second difference, giving the last difference zero; and thus the process ends.

The latter terms of the differences need not be expressed until the corresponding like terms in the partial products are to be subtracted from them; thus in the following ex-

ample $-17x+6$ is not expressed in the first difference, nor $+6$ in the second.

(2.) Divide $6x^4+5x^3+6x^2-17x+6$ by $2x-1$.

$$
\begin{array}{l}
2x-1)\ 6x^4+5x^3+6x^2-17x+6\ (3x^3+4x^2+5x-6 \\
\quad 6x^4-3x^3 \\
\hline
\quad\quad 8x^3+6x^2 \\
\quad\quad 8x^3-4x^2 \\
\hline
\quad\quad\quad 10x^2-17x \\
\quad\quad\quad 10x^2-5x \\
\hline
\quad\quad\quad\quad -12x+6 \\
\quad\quad\quad\quad -12x+6 \\
\hline\hline
\end{array}
$$

(3.) Divide $1-2x^3+x^6$ by $1-2x+x^2$.

$$
\begin{array}{l}
1-2x+x^2)\ 1-2x^3+x^6\ (1+2x+3x^2+2x^3+x^4 \\
\quad 1-2x+x^2 \\
\hline
\quad\quad 2x-x^2-2x^3 \\
\quad\quad 2x-4x^2+2x^3 \\
\hline
\quad\quad\quad 3x^2-4x^3 \\
\quad\quad\quad 3x^2-6x^3+3x^4 \\
\hline
\quad\quad\quad\quad 2x^3-3x^4 \\
\quad\quad\quad\quad 2x^3-4x^4+2x^5 \\
\hline
\quad\quad\quad\quad\quad x^4-2x^5+x^6 \\
\quad\quad\quad\quad\quad x^4-2x^5+x^6 \\
\hline\hline
\end{array}
$$

Here $+x^6$ is not expressed until we reach the last difference.

(4.) Divide $2a^5-6a^3b+13a^2b^2-6ab^3-3a^4b$ by $2a-3b$.

Here we shall arrange the dividend and divisor according to descending powers of a.

$$
\begin{array}{l}
2a-3b)\ 2a^5-3a^4b-6a^3b+13a^2b^2-6ab^3\ (a^4-3a^2b+2ab^2 \\
\quad 2a^5-3a^4b \\
\hline
\quad\quad -6a^3b+13a^2b^2 \\
\quad\quad -6a^3b+\ \ 9a^2b^2 \\
\hline
\quad\quad\quad 4a^2b^2-6ab^3 \\
\quad\quad\quad 4a^2b^2-6ab^3 \\
\hline\hline
\end{array}
$$

Exercise XIX.

Divide

(1.) $x^2-7x+12$ by $x-3$; and $3x^2+7x+2$ by $x+2$.

(2.) x^2-4x-5 by $x-5$; and $4x^2-9$ by $2x+3$.

(3.) $6x^2-5x-6$ by $2x-3$; and $9x^3-18x^2+26x-24$ by $3x-4$.

(4.) x^3-4x^2+5x-2 by x^2-3x+2; and x^4+x^2+1 by x^2+x+1.

(5.) $4x^4-15x^3+14x^2-6x+1$ by $4x^2-3x+1$.

(6.) x^4-1 by $x-1$; and x^5+1 by $x+1$.

(7.) $x^2-2xy+y^2$ by $x-y$; and x^3+y^3 by $x+y$.

(8.) $a^4+a^2b^2+b^4$ by a^2-ab+b^2.

(9.) $20x^2+9xy-12x-18y^2+9y$ by $4x-3y$.

(10.) $x^5-2ax^4+3a^2x^3-3a^3x^2+2a^4x-a^5$ by $x^3-ax^2+a^2x-a^3$.

(11.) $x^4-x^3y-xy^3+y^4$ by x^2+xy+y^2.

(12.) $\frac{2}{5}a^4x-\frac{136}{75}a^3x^2+\frac{8}{5}a^2x^3+\frac{3}{10}ax^4-x^5$ by $\frac{2}{3}a^3-\frac{4}{5}a^2x+\frac{1}{2}x^3$.

57. When the division is not exact, the last difference is called the remainder. In this case the product of the quotient and divisor added to the remainder will be equal to the dividend.

Example.

Find the quotient and remainder in dividing $10x^3+7x^2-8x-2$ by $2x+3$.

$$
\begin{array}{r|l}
2x+3)\ 10x^3+7x^2-8x-2 & (5x^2-4x+2 \\
\underline{10x^3+15x^2\quad\quad\quad\;\;} & \\
-8x^2-8x\quad\; & \\
\underline{-8x^2-12x\quad\;} & \\
4x-2 & \\
\underline{4x+6} & \\
-8 &
\end{array}
$$

Quotient $=5x^2-4x+2$.
Remainder $=-8$.

Exercise XX.

Find the quotient and remainder in dividing

(1.) $4x^3-4x^2+8x+2$ by $2x+1$.

(2.) x^2+a^2 by $x+a$.

(3.) x^3-a^3 by $x+a$.

(4.) $2x^5-2x^4+9x^3$ by $2x^3+x+1$.

(5.) $2x^5+2x^4+5x^3$ by x^3+x^2+x+1.

CHAPTER VII.

EXAMPLES INVOLVING THE APPLICATION OF THE FIRST FOUR RULES.

58. In the following examples some of the given quantities are expressed by letter symbols, and the object of the exercises is to express in like manner other quantities which by the conditions of the question are related to the former. When a doubt exists as to the manner of solving a question, it will be well to substitute numbers for letters in order to see what operations ought to be performed in the given symbols.

59. The sign $\therefore$ will be used to mean *hence*, or *therefore*, and the sign $\because$ *since*, or *because*.

Examples.

(1.) I buy goods for $2a+3b-c$ dollars, and sell them for $4a-b+2c$ dollars; what do I gain?

$$\begin{array}{r} 4a-b+2c \\ \underline{2a+3b-c} \\ 2a-4b+3c \end{array}$$

$\therefore$ the gain, which is the selling price less the cost price, is $2a-4b+3c$ dollars.

(2.) A man has $3x^2+7x+2$ dollars and spends $x+2$ of them per day; how long will his money last?

$$\begin{array}{l} x+2)\,3x^2+7x+2\,(3x+1 \\ \underline{3x^2+6x} \\ x+2 \\ \underline{x+2} \end{array}$$

$\therefore$ the required number of days, which is equal to the number of times the amount of his daily expenses is contained in the amount he possesses, is $3x+1$.

(3.) A man walks x miles in y hours: at what rate per hour does he walk; how far will he walk in 5 hours; and how long will he be in walking 12 miles?

$\because$ he walks x miles in y hours,

$\therefore$ „ „ $\frac{x}{y}$ „ „ 1 hour,

and 1 mile in $\frac{y}{x}$ hours.

Again, $\because$ he walks $\frac{x}{y}$ miles in 1 hour,

$\therefore$ „ „ $\frac{5x}{y}$ „ „ 5 hours;

and $\because$ he takes $\frac{y}{x}$ hours to walk 1 mile,

$\therefore$ „ „ $\frac{12y}{x}$ „ „ „ 12 miles.

The answers are, therefore, $\frac{x}{y}$ miles, $\frac{5x}{y}$ miles, and $\frac{12y}{x}$ hours.

Exercise XXI.

(1.) A man walks x, $x+a$, and $x-2a$ miles in the same direction; how far does he walk altogether?

(2.) A man has 100 dollars, and owes $50-x$ dollars; what is he worth?

(3.) A man walks $a+b$ miles and returns $a-b$ miles; how far is he from the starting point?

(4.) What is the area of a room $x+y$ feet long and $x-y$ feet broad?

(5.) A man walks x miles at a miles an hour; how long is he on the road?

(6.) A has x dollars, B $50y$ cents, and C $75z$ cents; how many dollars have A, B, and C together?

(7.) A has x pounds, B has y shillings, and C z pence; how many pounds have A, B, and C together?

(8.) A spends a dollars in x days; in how many days will he spend 10 dollars?

(9.) How many square yards in a floor which is a feet by x feet?

(10.) What is the cost in dollars of painting a floor $x+y$ feet by $x-y$ feet at x^2+y^2 cents per square foot?

(11.) A owns a acres, B b acres, and C 5 acres less than one-half of A's and one-third of C's together; what is the whole amount possessed by A, B, and C?

(12.) A owns $a+b$ acres, B $a-b$ acres, C half as much as A, and D half as much as B; how much more do A and C own than B and D?

(13.) A walks a miles in t hours, and B half as far again in the same time; how far will B walk in 10 hours?

(14.) A walks 10 miles in x hours; how long will he be in walking a miles?

(15.) A spends at the rate of x dollars a day for a days, a dollar a day less for twice that time, and a dollar a day more for three times that time; how much does he spend altogether?

CHAPTER VIII.

SIMPLE EQUATIONS.

60. An *equation* is the statement of the equality of different quantities, and these quantities are called the equation's *members* or *sides*.

Thus $2x+3=7$ is an equation whose sides are $2x+3$ and 7, and $x^2-5x+6=0$ is an equation whose sides are x^2-5x+6 and 0.

61. An *identity* is the statement of the equality of two like or different forms of the same quantity.

Thus $2a+b=2a+b$, $2x+3x=5x$, $x^2-5x+6=(x-2)(x-3)$, are identities.

62. In the case of an identity the equality holds for all values of the quantities involved, whereas in an equation the equality does not exist except for a limited number of values of the quantities involved.

Thus the statement $x^2+2x+1=(x+1)^2$ holds no matter what x is; but $5x-3=7$ holds only when $x=2$, and $x^2+6=5x$ only when $x=2$, or $x=3$.

63. A symbol to which a particular value or values must be assigned in order that the statement contained in an equation may be true is called an *unknown* quantity.

Thus the unknown quantity in $5x-6=9$ is x, and in $2y^2-y=8$ is y.

The letters a, b, c, l, m, n, p, q, r are generally used to denote quantities which are supposed to be known, and x, y, z those which are for the time unknown.

64. Quantities which on being substituted for the unknown reduce the equation to an identity are said to *satisfy* the equation, and are called its *roots*.

Thus 5 is a root of $2x-3=7$, because 5 when substituted for x reduces the equation to the identity $10-3=7$. So 2 and 3 are the roots of $x^2+6=5x$, because when either is substituted for x the equation is satisfied.

65. The determination of the root or roots is called the *solution* of the equation.

66. An equation is said to be reduced to its simplest form when its members consist of a series of mononomials involving positive integral powers only of the unknown.

Thus $5x-8=0$, $x^2-5x+8=0$, $2x^2+6x=7$, $x^3-6x^2=7x-8$ are in their simplest forms.

67. Equations when reduced to their simplest forms are classed according to their *order* or *degree*.

68. *Simple* equations, or those of the first degree, are those in which the highest power of the unknown quantity is the first; as, for example, $2x=5$, $5x-8=0$, $3x-7=0$.

69. *Quadratic* equations, or those of the second degree, are those in which the highest power of the unknown quantity is the second; as, for example, $x^2-2x+3=0$, $2x^2=9$, $4x^2-3=10x$.

70. Equations of the third and fourth degrees are called *cubic* and *biquadratic* equations, respectively; thus $x^3+2x=10$ is a cubic, and $x^4-2x^3=10x-5$ a biquadratic.

71. It is proved in works on the *Theory of Equations* that the number of the roots of an equation is equal to its degree; so that a simple equation has one root, a quadratic two roots, a cubic three roots, and so on.

72. In order to solve an equation it is generally necessary to reduce it by one or both of the following processes:—

I. Transposition of Terms.

II. Clearing of Fractions.

These operations will be illustrated by applying them in order to the solution of simple equations.

I. Transposition of Terms.

73. If an equation contains no fractions it may be solved by *transposition of terms*, which *consists in taking the unknown quantities to one side of the equation and the known to the other, the signs of the quantities which are so transposed being changed.*

Thus, if the equation is $4x+5=10$, by subtracting 5 from each side we get

$$4x+5-5=10-5,$$

or $$4x=5;$$

and so any quantity may be transposed from one side to the other by changing its sign.

Examples.

(1.) Solve $5x+15=25$.

Transposing $+15$ we get

$$5x=25-15=10.$$

The value of x is then found by dividing both sides by its coefficient 5.

$$\therefore x=2.$$

(2.) Solve $8x-4=2x+20$.

Transposing -4, $8x=2x+20+4$.

Transposing $2x$, $8x-2x=20+4;$

$$6x=24.$$

$$\therefore x=4.$$

(3.) Solve $10+2(6x-1)=32-3(x-4)$.

Clearing of brackets,

$$10+12x-2=32-3x+12.$$

Transposing $10, -2, -3x$,

$$12x+3x=32+12-10+2;$$

$$15x=36.$$

$$\therefore x=\tfrac{36}{15}=2\tfrac{2}{5}.$$

(4.) Solve $3(x^2+2x)+13=3x^2-7+4(3x-1)$.

Clearing of brackets,

$$3x^2+6x+13=3x^2-7+12x-4.$$

Transposing $+13$, $3x^2$, $+12x$,

$$3x^2-3x^2+6x-12x=-13-7-4;$$
$$-\ 6x=-24.$$

Dividing by -6,

$$x=4.$$

When, as in this case, the same quantity is common to both sides, it may be struck out without actually transposing; thus

$$5x^2-6x+7=8x+5x^2-10$$

becomes $$-6x+7=8x-10.$$

(5.) Solve $$ax+b=c.$$

Transposing $+b$, $$ax=c-b.$$

Dividing by a, $$x=\frac{c-b}{a}.$$

(6.) Solve $$ax+b=cx+d.$$

Transposing $+b$, cx,

$$ax-cx=d-b.$$

Collecting coefficients of x,

$$(a-c)x=d-b.$$

Dividing by $a-c$,

$$x=\frac{d-b}{a-c}.$$

(7.) Solve $$a(x-b)=b(x+a)-c.$$

Clearing of brackets,

$$ax-ab=bx+ab-c.$$

Transposing $-ab$, bx,

$$ax-bx=ab+ab-c.$$

Collecting coefficients of x,

$$(a-b)x=2ab-c.$$

Dividing by $a-b$,

$$x=\frac{2ab-c}{a-b}.$$

Exercise XXII.

(1.) $3+x=5.$ (2.) $x-6=4.$ (3.) $x+5=12.$
(4.) $x+9=4.$ (5.) $2x-1=3.$ (6.) $5x+4=29.$
(7.) $4-3x=5.$ (8.) $1-x=6.$ (9.) $3=6-2x.$
(10.) $2x+3=x+5.$ (11.) $5x-2=2x+7.$
(12.) $x+4=18-4x-4.$ (13.) $2x+3=3x-4.$
(14.) $16-2x=46-5x.$ (15.) $3(x-1)+4=4(4-x).$
(16.) $5-3(4-2x)+4(3-4x)=0.$
(17.) $x-1-2(x-2)+3(x-3)=6.$
(18.) $5(x-5)+2(x-3)-(x-1)=9.$
(19.) $2(x-2)-3(x-3)+4(x-4)-5(x-5)=0.$
(20.) $4(x-11)-7(x-12)=6-(x-8).$
(21.) $x=2a-x.$ (22.) $2a-3x=8a-5x.$
(23.) $x-2b=2a-x.$ (24.) $a+x-b=a+b.$
(25.) $ax-ab-ac=0.$ (26.) $ax-a=b-bx.$
(27.) $ax-a^3=bx-b^3.$ (28.) $a(x-b)=c(x-a).$

II. Clearing of Fractions.

74. If an equation contains fractions, it may be reduced to a form capable of solution by transposition, *by multiplying both sides of the equation by the L. C. M. of all the denominators of the fractions.*

In the following examples numerical denominators only will be considered.

(1.) Solve $\frac{x}{2}-\frac{x}{3}=\frac{x}{5}-1.$

Multiplying by 30, the L.C.M. of 2, 3, 5,

$$15x-10x=6x-30.$$

Transposing, $15x-10x-6x=-30;$

$$-x=-30.$$

$$\therefore\ x=30.$$

(2.) Solve $\frac{x-1}{2}+\frac{2x+3}{3}=\frac{6x+19}{8}.$

Multiplying by 24, the L.C.M. of 2, 3, 8,

$$12(x-1)+8(2x+3)=3(6x+19);$$

whence on clearing of brackets and transposing we get

$$x=4\tfrac{1}{2}.$$

It must be observed that $\dfrac{x-1}{2}=\tfrac{1}{2}(x-1)$, $\dfrac{2x+3}{3}=\tfrac{1}{3}(2x+3)$, and $\dfrac{6x+19}{8}=\tfrac{1}{8}(6x+19)$; and therefore the brackets must be supplied in the first step since the numerators become binomial factors.

(3.) Solve $\qquad 3-\dfrac{x-1}{2}+\dfrac{x+2}{3}=0.$

Multiplying by 6, the L.C.M. of 2 and 3,

$$18-3(x-1)+2(x+2)=0;$$

whence on clearing of brackets and transposing we get

$$x=25.$$

Exercise XXIII.

(1.) $\dfrac{x}{2}-\dfrac{x}{4}=3.$ (2.) $\dfrac{x}{3}+\dfrac{x}{4}=7.$ (3.) $\dfrac{x}{2}-\dfrac{x}{3}+\dfrac{x}{4}=10.$

(4.) $\dfrac{x-6}{4}+\dfrac{x-6}{6}=10.$ (5.) $\dfrac{x-3}{2}+\dfrac{x}{3}=20-\dfrac{x-19}{2}.$

(6.) $\dfrac{x}{2}+\dfrac{x-3}{3}=4-\dfrac{x-8}{4}.$ (7.) $\tfrac{1}{2}x+2=x-\tfrac{1}{7}(x+1).$

(8.) $2(x-1)-\tfrac{1}{2}(2x-9)=\tfrac{1}{2}(17-2x).$

(9.) $\dfrac{8x-9}{5}=4-\dfrac{5x-12}{3}.$ (10.) $\tfrac{2}{5}(x-4)+\tfrac{1}{3}(x-6)=2x-15.$

(11.) $3-\tfrac{2}{3}x=1-\tfrac{1}{15}(7x-18).$

(12.) $6(x-1)-1=\tfrac{6}{5}(5-2x)+\tfrac{3}{2}(x+1)+4x.$

(13.) $x+\dfrac{4x+7}{3}-\dfrac{2x+2}{9}-9\tfrac{1}{2}=0.$

(14.) $4x+2\tfrac{1}{21}+\dfrac{2(7-4x)}{7}=\dfrac{8x}{7}+\tfrac{13}{21}.$

(15.) $\dfrac{x+4}{2}-\dfrac{3x-2}{12}+\tfrac{1}{4}=\dfrac{x-1}{3}.$

(16.) $\dfrac{8-2x}{14}+\dfrac{6x}{7}-\tfrac{5}{7}=\dfrac{3(2x+6)}{14}-\dfrac{2x}{7}.$

(17.) $\frac{3x-1}{5}-\frac{13-x}{2}-\frac{7x}{3}+\frac{11x+33}{6}=0.$

(18.) $\frac{1-x}{4}+\frac{2-x}{5}+\frac{3-x}{6}=\frac{x}{3}-\frac{3}{4}.$

(19.) $\frac{x}{2}-\frac{2x}{3}+\frac{3x}{4}-\frac{4x}{5}+\frac{2(x-3)}{9}=0.$

(20.) $\frac{2}{5}(x-3)-\frac{x-4}{9}-\frac{3x}{13}=0.$

CHAPTER IX.

PROBLEMS.

75. When a question is assigned for solution the unknown quantity, or number, is generally involved in the various conditions which are proposed for its determination. The expression of these conditions in Algebraical language leads to an equation, the solution of which will be the solution of the question.

76. In some cases, although there are more unknowns than one, they are related to each other in such a manner that when one is determined the others become immediately known. In such cases the unknowns can be expressed in terms of one unknown.

Thus, if the sum of two unknowns is equal to 8, we may denote one of them by x and the other by $8-x$, or one of them by $4+x$ and the other by $4-x$; if the greater of two unknowns exceeds the less by 3, the former may be denoted by x, and the latter by $x-3$; if there be two numbers of which one exceeds 4 times the other by 7, the former may be denoted by $4x+7$, and the latter by x.

In like manner, if there be three unknowns, of which the first exceeds the second by 3, and the second exceeds the third by 5, the first may be denoted by x, the second by $x-3$, and the third by $x-8$.

77. The following examples will illustrate the method of solving problems by means of simple equations :—

(1.) What number exceeds its fifth part by 20?

Let x be the required number.

Then its fifth part $=\frac{x}{5}$; and by the condition of the question

$$x-\frac{x}{5}=20.$$

$$\therefore\ x=25.$$

(2.) The sum of two numbers is 71, and their difference 43: find them.

Let x be the greater number.

Then $x-43$ is the less: and since their sum is 71, we have

$$x+x-43=71.$$

$$\therefore\ x=57,\text{ the greater;}$$

$$\text{and } x-43=14,\text{ the less.}$$

This question may also be solved as follows:

Let x be one number, the greater suppose.

Then $71-x$ is the less; and since their difference is 43, we have

$$x-(71-x)=43.$$

$$\therefore\ x=57,$$

$$\text{and } 71-x=14.$$

(3.) A boy is one-third the age of his father, and has a brother one-sixth of his own age; the ages of all three amount to 50 years. Find the age of each.

Let the boy's age $=x$ years.

Then the father's age $=3x$ years,

And the brother's age $=\frac{x}{6}$ years.

And by the condition of the question

$$x+3x+\frac{x}{6}=50.$$

$$\therefore\ x=12,$$

$$3x=36,$$

$$\frac{x}{6}=2.$$

Fractions may be avoided by supposing the ages of boy, father, and brother to be $6x$, $18x$, x years, respectively.

(4.) A and B start from two places, 90 miles apart, at the same moment, A walking 4 miles per hour, and B 5; when will they meet, and how far will each have walked?

Let the time of meeting be x hours after starting.

Then A will have walked $4x$ miles, and B $5x$ miles; and since the sum of these two distances is 90 miles,

$$4x+5x=90.$$
$$\therefore\ x=10.$$

$\therefore$ $4x=40$, and $5x=50$, are the distances in miles walked by A and B, respectively.

(5.) How much tea at 90 cents per lb. must be mixed with 50 lbs. at \$1·20, that the mixture may be sold at \$1·10?

Let x = the number of lbs. at 90 cents, the value of which will be $\cdot 90x$ dollars.

Then, since there will be $x+50$ lbs. in the mixture, its value will be $1\cdot 10\,(x+50)$ dollars; and since the value of the 50 lbs. at \$1·20 is 60 dollars, we have

$$\cdot 90x+60=1\cdot 10(x+50).$$

Multiplying by 100,

$$90x+6000=110(x+50).$$
$$\therefore\ x=25.$$

Exercise XXIV.

(1.) Divide 25 into two such parts that 6 times the greater exceeds twice the less by 70.

(2.) Divide 135 into two parts such that one shall be $\frac{4}{5}$ the other.

(3.) The sum of two numbers is 37 and their difference 3: find them.

(4.) A fish weighed 7lbs. and half its weight: how much did it weigh?

(5.) At a meeting 43 members were present, and the motion was carried by 9: how many voted on each side?

(6.) Divide 326 into two parts, such that $\frac{6}{5}$ of the one shall be equal to the other diminished by 7.

(7.) What is the number whose 4th and 5th parts added together make $2\frac{1}{4}$?

(8.) Forty-two years hence a boy will be 7 times as old as he was 6 years ago: how old is he?

(9.) A father is 57 years old, his son 13: when will the father be 3 times as old as his son?

(10.) I have made 164 runs at cricket this season in 12 innings: how many must I make in my next innings to average 14?

(11.) My grandfather told me 10 years ago that he was 7 times as old as myself; I am now 18: how old is my grandfather?

(12.) If in a theatre $\frac{3}{8}$ of the seats are in the pit, $\frac{3}{10}$ in the lower gallery, $\frac{1}{5}$ in the upper, and there are 50 reserved seats, how many are there altogether?

(13.) After losing $\frac{1}{5}$ of our men by sickness, and 210 killed and wounded, the regiment was reduced by $\frac{1}{2}$: how many men did the regiment originally contain?

(14.) In a certain examination $\frac{3}{4}$ of a boy's marks were gained by translation, $\frac{1}{8}$ by mathematics, and $\frac{1}{10}$ by Latin prose: he also obtained 1 mark for French. How many marks did he obtain for each subject?

(15.) Two men receive the same sum; but if one were to receive 15 shillings more, and the other 9 shillings less, the one would receive 3 times as much as the other. What sum did they receive?

(16.) A and B begin trade, A with 3 times as much stock as B. They each gain £50, and then 3 times A's stock is exactly equal to 7 times B's. What were their original stocks?

(17.) One-tenth of a rod is coloured red, one-twentieth orange, one-thirtieth yellow, one-fortieth green, one-fiftieth

blue, one-sixtieth indigo, and the remainder, which is 302 inches long, white: what is its length?

(18.) Find three numbers whose sum is 37, such that the greater exceeds the second by 7, and the second exceeds the third by 3.

(19.) Find a number such that if 5, 11, and 17 be successively subtracted from it, the sum of the third, fourth, and sixth parts of the respective results shall be equal to 19.

(20.) How much wheat at 44s. a quarter must be mixed with 120 quarters at 60s. that the mixture may be sold for 50s. a quarter?

(21.) How many lbs. of tea at 2s. 6d. per lb. must be mixed with 18 lbs. at 5s. per lb. that the mixture may be sold for 4s. per lb.?

(22.) How much sugar at $4\frac{1}{2}$d. per lb. must be mixed with 50 lbs. at $6\frac{1}{2}$d. per lb., that the mixture may be worth 5d. per lb.?

(23.) A bag contains a certain number of sovereigns, twice as many shillings, and three times as many pence; and the whole sum is £267; find the number of sovereigns, shillings, and pence.

(24.) I wish to divide £5 4s. into the same number of crowns, florins, and shillings; how many coins must I have of each sort?

(25.) A person gets an income of £550 a year from a capital of £13,000, part of which produces 5 per cent. and part 4 per cent.: what are the amounts producing 5 and 4 per cent., respectively?

(26.) I invest £800, partly at $4\frac{1}{2}$ per cent., and partly at $5\frac{1}{2}$ per cent.; my income is £39 10s.: what are the sums invested at $4\frac{1}{2}$ and $5\frac{1}{2}$ per cent., respectively?

(27.) A garrison consists of 2600 men, of whom there are 9 times as many infantry and 3 times as many artillery as there are cavalry: how many men are there of each?

(28.) My grandfather's age is 5 times my own; if I had

been born 100 years ago, I should have been born 15 years before my grandfather: how old am I?

(29.) There is a number consisting of two figures of which the figure in the unit's place is 3 times that in the ten's; if 36 be added, the sum is expressed by the digits reversed: what is the number?

(30.) A miner works for 6 weeks (exclusive of Sundays), his wages being at the rate of 24s. per week, but he is to forfeit 1s. besides his pay for each day that he is absent; at the end of the time he receives 4 guineas: how many days was he absent?

(31.) A contractor finds that if he pays his workmen 2s. 6d. per day, he will gain 10s. per day on the job; if he pays them 3s. a day, he will lose 18s.: how many workmen are there, and what does the contractor receive per day?

(32.) An officer on drawing up his men in a solid square finds he has 34 men to spare, but increasing the side by 1 man he wants 39 to make up the square: how many men had he?

(33.) If the mean velocity of a cannon-ball at effective ranges is 1430 feet per second, and that of sound 1100 feet, how far is a soldier from a fort who hears the report of a gun $\frac{9}{10}$ of a second after he is hit?

(34.) An army in a defeat loses one-sixth of its number in killed and wounded, and 4000 prisoners. It is reinforced by 3000 men; but retreats, losing a fourth of its number in doing so. There remain 18,000 men. What was the original force?

(35.) Suppose the distance between London and Edinburgh is 360 miles, and that one traveller starts from Edinburgh and travels at the rate of 10 miles an hour, while another starts at the same time from London and travels at the rate of 8 miles an hour: it is required to know where they will meet.

(36.) There are two places 154 miles apart, from which two persons start at the same time with a design to meet; one travels at the rate of 3 miles in 2 hours, and the other at the rate of 5 miles in 4 hours: when will they meet?

CHAPTER X.

PARTICULAR RESULTS IN MULTIPLICATION AND DIVISION.

78. There are several results in multiplication and division which should be committed to memory, as they enable us to dispense with the labour of performing the operations. The following cases occur most frequently.

I. Since by actual multiplication

$$\begin{aligned}(a+b)^2 &= a^2+b^2+2ab,\\ (a-b)^2 &= a^2+b^2-2ab,\\ (a+b-c)^2 &= a^2+b^2+c^2+2ab-2ac-2bc,\\ \&c. &= \&c.\end{aligned}$$

we can hence write down the square of a polynomial by the rule:

The square of a polynomial is equal to the sum of the squares of the several terms and twice the sum of the products of every two terms.

Thus in the last example a^2, $+b^2$, $+c^2$, are, respectively, the squares of a, $+b$, $-c$; $+2ab$ is twice the product of a and $+b$, $-2ac$ is twice the product of a and $-c$, and $-2bc$ is twice the product of $+b$ and $-c$.

In taking the products of the terms, two and two, it will be found most convenient to take in order the products of the first term and every term that follows it, then the products of the second term and every term that follows it, and so on, if there be more terms than three.

Examples.

(1.) $(a+2x)^2=a^2+4x^2+4ax.$

Here $+4ax$ is twice the product of a and $+2x$.

(2.) $(2a-5x)^2=4a^2+25x^2-20ax.$

Here $+25x^2$ is the square of $-5x$, and $-20ax$ is twice the product of $2a$ and $-5x$.

(3.) $$\begin{aligned}(2x^2-3x+4)^2&=4x^4+9x^2+16-12x^3+16x^2-24x\\&=4x^4-12x^3+25x^2-24x+16.\end{aligned}$$

Here $-12x^3$ is twice the product of $2x^2$ and $-3x$, $+16x^2$ of $2x^2$ and 4, and $-24x$ of $-3x$ and $+4$. Like terms are added together and the terms are arranged according to descending powers of x.

(4.) $99^2=(100-1)^2=10000+1-200=9801.$

Exercise XXV.

Write down the squares of

(1.) $x-1$, $x+a$, $x-5$, $x+3$.

(2.) $2x+1$, $3x-1$, $2x+3$, $3x-2$.

(3.) x^2-a, $2xy+1$, $3x^2-2a$, ax^2-4b.

(4.) $x-y+z$, $2x+3y-z$, $x-2y-5z$, $2x-4y+1$.

(5.) $2a^2+a+3$, $3a^2-4a+1$, a^2-2a-4.

(6.) Find the squares of 49, 98 and 995.

79. II. Since $(a+b)(a-b)=a^2-b^2$, it follows that *the product of the sum and difference of two quantities is equal to the difference of their squares.*

Thus $$\begin{aligned}(2x+3y)(2x-3y)&=4x^2-9y^2;\\(a^2+1)(a^2-1)&=a^4-1;\\(5x^2+4y)(5x^2-4y)&=25x^4-16y^2;\\(2x^3+a^4)(2x^3-a^4)&=4x^6-a^8;\\501\times499&=(500+1)(500-1)\\&=500^2-1\\&=249999.\end{aligned}$$

Exercise XXVI.

Write down the products of

(1.) $x-1, x+1$; $a+3, a-3$; $2+x, 2-x$.

(2.) $2x+1, 2x-1$; $5a+2, 5a-2$; $4x+a, 4x-a$.

(3.) a^2+x, a^2-x; a^3+1, a^3-1; a^5+x^2, a^5-x^2.

(4.) $3a^2+2b$, $3a^2-2b$; $4a^3+2x^2$, $4a^3-2x^2$; $7a^4-5a^3$, $7a^4+5a^3$.

(5.) Find the products of 48, 52; 95, 105; 695, 705.

80. III. Since by actual multiplication

$$(a^2+b^2-ab)(a+b)=a^3+b^3,$$
$$(a^2+b^2+ab)(a-b)=a^3-b^3,$$

it follows that *the sum of the squares less the product of two quantities multiplied by their sum is equal to the sum of their cubes.*

In the latter identity the two quantities are a and $-b$; the sum of their squares less their product is, therefore, $a^2+b^2--ab=a^2+b^2+ab$; and, since the cube of $-b$ is $-b^3$, the sum of their cubes is a^3-b^3.

Examples.

(1.) $(x^2-x+1)(x+1)=x^3+1$.

Here the two quantities are x and 1.

(2.) $(x^2+x+1)(x-1)=x^3-1$.

Here the two quantities are x and -1.

(3.) $(4x^2-2x+1)(2x+1)=8x^3+1$.

Here the two quantities are $2x$ and 1, the cubes of which are $8x^3$ and 1.

(4.) $(x^4-a^2x^2+a^4)(x^2+a^2)=x^6+a^6$.

Here the two quantities are x^2 and a^2, the cubes of which are x^6 and a^6.

(5.) $(4x^4+6x^2y+9y^2)(2x^2-3y)=8x^6-27y^3$.

Here the two quantities are $2x^2$ and $-3y$, the cubes of which are $8x^6$ and $-27y^3$.

Exercise XXVII.

Write down the products of

(1.) $m^2-mn+n^2,\ m+n;\ p^2+pq+q^2,\ p-q.$

(2.) $m^2-m+1,\ m+1;\ 1+q+q^2,\ 1-q.$

(3.) $x^2-3x+9,\ x+3;\ a^2+4a+16,\ a-4.$

(4.) $4a^2-2a+1,\ 2a+1;\ 16x^2+4ax+a^2,\ 4x-a.$

(5.) $4a^2-6ab+9b^2,\ 2a+3b;\ 9x^2+15xy+25y^2,\ 3x-5y.$

(6.) $x^4+x^2+1,\ x^2-1;\ x^6-a^2x^3+a^4,\ x^3+a^2.$

81. IV. By actual division it can be shown that *the sum of any the same odd powers of two quantities is exactly divisible by the sum of the quantities.*

Thus,

$$\frac{x+y}{x+y}=1,$$

$$\frac{x^3+y^3}{x+y}=x^2-xy+y^2,$$

$$\frac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4,$$

&c.=&c.

It will be observed that the signs of the quotient are alternately + and −, and that the successive powers of x are in descending whilst those of y are in ascending order.

82. V. *The difference of any the same odd powers of two quantities is exactly divisible by the difference between the quantities.*

Thus,

$$\frac{x-y}{x-y}=1,$$

$$\frac{x^3-y^3}{x-y}=x^2+xy+y^2,$$

$$\frac{x^5-y^5}{x-y}=x^4+x^3y+x^2y^2+xy^3+y^4,$$

&c.=&c.

Here the signs of the quotient are all $+$.

It may be noted that this case is included in the preceding (81) by supposing the two quantities to be x and $-y$. Thus the sum of the cubes of x and $-y$ is x^3-y^3, which is exactly divisible by their sum $x-y$. In fact the formulas of (82) are deducible from those of (81) by substituting $-y$ for y in the latter.

83. VI. *The difference between any the same even powers of two quantities is exactly divisible by the sum of the quantities and also by their difference.*

Thus (i.) $$\frac{x^2-y^2}{x+y}=x-y,$$

$$\frac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3,$$

$$\frac{x^6-y^6}{x+y}=x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5,$$

&c.=&c.

(ii.) $$\frac{x^2-y^2}{x-y}=x+y,$$

$$\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3,$$

$$\frac{x^6-y^6}{x-y}=x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5,$$

&c.=&c.

It will be observed that when the divisor is $x-y$, the signs of the quotient are all $+$; and when the divisor is $x+y$ the signs of the quotient are alternately $+$ and $-$. It should also be noted that the formulas (ii.) are deducible from (i.) by substituting $-y$ for y in the latter.

Exercise XXVIII.

Write down the quotients of

(1.) x^3+1 and x^5+1 divided by $x+1$.

(2.) x^3-1 and x^5-1 divided by $x-1$.

(3.) x^2-1 and x^4-1 divided by $x+1$.

(4.) x^2-1 and x^4-1 divided by $x-1$.

(5.) $4a^2-9b^2$ divided by $2a+3b$.

(6.) $9x^6-4a^2$ divided by $3x^3-2a$.

(7.) $\frac{1}{4}a^4-x^6$ divided by $\frac{1}{2}a^2+x^3$.

(8.) Find what the quotient of x^3+y^3 divided by $x+y$ becomes when (i.) $x=2a$, $y=3b$; (ii.) $x=a^2$, $y=2$.

(9.) Find what the quotient of x^3-y^3 divided by $x-y$ becomes when (i.) $x=3a$, $y=b$; (ii.) $x=2a^2$, $y=3b$.

CHAPTER XI.

INVOLUTION AND EVOLUTION.

84. THE process by which the powers of quantities are expressed as mononomials is called *Involution.* The powers of polynomials when so expressed are said to be *developed*, or *expanded.*

85. We have already explained the notation for denoting the powers of a single symbol, as a, x, y. In all other cases the power of a quantity is denoted by enclosing it in brackets with the number indicating the power above and to the right of the bracket.

Thus $(-2a)^3$ denotes the third power of $-2a$; $(a^2b)^2$ the square of a^2b; $(a^3bc^5)^4$ the fourth power of a^3bc^5; $(a-b)^3$ the cube of $a-b$; $(x^2-2x+3)^5$ the fifth power of x^2-2x+3.

86. The same notation is used for denoting powers of powers of a quantity, brackets of different shapes being employed when necessary.

Thus $(a^3)^2$ denotes the square of a^3; $\{(-2xy)^2\}^3$ the cube of $(-2xy)^2$; $\{(x^2-5)^3\}^4$ the fourth power of $(x^2-5)^3$.

EXERCISE XXIX.

Retaining the given quantities, denote

(1.) The cubes of $-a$, $2x$, $3xy^2$, $2a^4b^2c$.

(2.) The squares of $2a-1$, $a-b+1$, x^3-1.

(3.) The squares of $(x^3)^4$, $(-2a)^3$, $(4ax)^5$, $(3a^2bc^4)^3$.

(4.) The cubes of $(a-b)^2$, $(x^2-1)^4$, $(x^2-3x+2)^2$.

(5.) The squares of the cubes of x, $-2x$, a^2b, $x-a$, x^2-ax+1.

(6.) The cubes of the squares of $-a$, x^2, $4x-1$, x^3-a^3.

87. A power of a power of a quantity is expressed as a power of that quantity according to the rule

$$(a^m)^n=a^{mn}.$$

Thus,
$$(a^2)^3=a^2\cdot a^2\cdot a^2=a^6;$$
$$(a^3)^2=a^3\cdot a^3=a^6;$$
$$(a^3)^4=a^3\cdot a^3\cdot a^3\cdot a^3=a^{12};$$
$$(a^4)^3=a^{12};$$
$$(a^5)^3=a^{15};$$
$$\{(a^3-b)^2\}^3=(a^3-b)^6.$$

88. So also
$$(a^m b^p)^n=a^{mn}\cdot b^{pn},$$
$$(a^m b^p c^q)^n=a^{mn}\cdot b^{pn}\cdot c^{qn}, \text{ \&c.}$$

Thus,
$$(a^2b)^3=a^2b\cdot a^2b\cdot a^2b=a^6b^3;$$
$$(a^2b^3c^4)^3=a^6b^9c^{12};$$
$$(ab^3c^5)^4=a^4b^{12}c^{20}.$$

89. A rule has already been given in Art. 78 for expanding the square of any polynomial. The expansion may also be effected as in the following examples, in which the various parts are arranged in rows. *In the first row occurs the square of the first term of the given quantity; in the second row the product of twice the first term added to the second and the second; in the third row the product of twice the first term added to twice the second term added to the third term and the third; and so on.*

Examples.

(1.) $(a+b)^2=a^2$
$\qquad\qquad +(2a+b)\,b=\text{\&c.}$

(2.) $(b-c)^2=b^2$
$\qquad\qquad +(2b-c)\,(-c)=\text{\&c.}$

(3.) $(a+b+c)^2=a^2$
$\quad +(2a+b)b$
$\quad +(2a+2b+c)c=\&c.$

(4.) $(a-b-c)^2=a^2$
$\quad +(2a-b)(-b)$
$\quad +(2a-2b-c)(-c)=\&c.$

(5.) $(a^2-b+c^2-d)^2=(a^2)^2$
$\quad +(2a^2-b)(-b)$
$\quad +(2a^2-2b+c^2)c^2$
$\quad +(2a^2-2b+2c^2-d)(-d)=\&c.$

Exercise XXX.

Express as powers or products of powers

(1.) $(x^2)^3$, $(2x^2)^3$, $(x^3)^3$, $(2x^3)^3$, $(3x^2)^4$.
(2.) $(ax^2)^2$, $(a^2x^3)^2$, $(a^2x^3y^4)^2$.
(3.) $(abc^2)^3$, $(a^2bc^2)^3$, $(2a^2b^3c^4)^3$.
(4.) $(x^2y^3z^4)^5$, $(ab^4c^7)^6$.

Expand

(5.) $(x+1)^2$, $(2x-3)^2$, $(x^2-5)^2$, $(x^3-2a^2)^2$.
(6.) $(x^2+2x+3)^2$, $(x^2-3x+4)^2$, $(2x^3-x^2+5)^2$.

90. Higher powers of polynomials are developed by the *Binomial* and *Multinomial Theorems,* the explanation of which may be found in more advanced works.

91. The process by which the roots of quantities are determined is called *Evolution.*

92. The nth root of a quantity is denoted by writing the quantity under the sign $\sqrt[n]{\quad}$, the line above being sometimes replaced by brackets enclosing the given quantity.

Thus, $\sqrt{2a}$ denotes the square root of $2a$;
$\sqrt[3]{5a^2}$ „ cube „ $5a^2$;
$\sqrt[4]{x^2+3}$ „ fourth „ x^2+3;
$\sqrt[5]{}(a^2-2a+3)$ denotes the fifth root of a^2-2a+3.

93. The mth root of the nth root of a quantity is denoted by writing the quantity under the sign $\sqrt[m]{\sqrt[n]{\quad}}$.

Thus, $\sqrt[3]{\sqrt{2a}}$ denotes the cube root of $\sqrt{2a}$;

$\sqrt[4]{\sqrt[3]{5a^2}}$ „ fourth „ $\sqrt[3]{5a^2}$;

$\sqrt[3]{\sqrt{x^5-x^3+1}}$ „ cube „ $\sqrt{x^5-x^3+1}$.

94. The mth root of the nth root of a quantity is expressed as a root of that quantity by the rule

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

Thus, $\sqrt[3]{\sqrt{a}} = \sqrt[6]{a}$; $\sqrt[3]{\sqrt[4]{2x}} = \sqrt[12]{2x}$;

$\sqrt[4]{\sqrt[5]{2x^3-1}} = \sqrt[20]{2x^3-1}$.

95. The reason of this rule will appear from the following case:—

Let $\sqrt[3]{\sqrt{a}}=x$. Then on cubing both sides of this equation we have

$$\sqrt{a}=x^3.$$

Squaring $a=x^6$.

Extracting the sixth root,

$$\sqrt[6]{a}=x=\sqrt[3]{\sqrt{a}}.$$

Exercise XXXI.

Retaining the given quantities, denote

(1.) The square roots of $2x$, ax^2, x^2-1, x^2-3x+4.

(2.) The cube roots of $-x^6$, $3a^3$, $a-b$, (a^3-3a+4).

(3.) The square roots of the cube roots of $3ax$, $x-1$, x^6+1.

(4.) The fourth roots of the cube roots of 2, $3x-1$, $2a^4-a^2+3$.

Express as roots of the quantities under the double sign

(5.) $\sqrt{\sqrt{a}}$, $\sqrt{\sqrt[3]{2x}}$, $\sqrt[3]{\sqrt{3x}}$, $\sqrt[3]{\sqrt[4]{2a}}$.

(6.) $\sqrt{\sqrt[3]{x^4-1}}$, $\sqrt[4]{\sqrt[3]{2x^3-5}}$, $\sqrt[5]{\sqrt[4]{x^8-6x^4+7}}$.

96. Since the square of a quantity is equal to the square of the same quantity with its sign or signs changed, it follows that there will be two square roots (if there be any), one being derived from the other by a change of signs.

Thus, since $(+a)^2=(-a)^2=+a^2$, it follows that the square root of $+a^2$ is $+a$ or $-a$. These two roots may be represented by the symbol $\pm a$ (read *plus or minus* a); so that we have $\sqrt{a^2}=\pm a$; $\sqrt{x^4}=\pm x^2$; $\sqrt{9x^2}=\pm 3x$.

Again, since by Art. 78

$$(a-b)^2=(-a+b)^2=a^2-2ab+b^2,$$
$$(a-b+c)^2=(-a+b-c)^2=a^2+b^2+c^2-2ab+2ac-2bc,$$
$$\&c.=\&c.,$$

it follows that

$$\sqrt{a^2-2ab+b^2}=a-b, \text{ or } -a+b,$$
$$=\pm(a-b);$$
$$\sqrt{(a^2+b^2+c^2-2ab+2ac-2bc)}=a-b+c, \text{ or } -a+b-c$$
$$=\pm(a-b+c).$$

In the following examples we shall only determine that square root of a polynomial whose leading term is $+$, the other being derivable by a mere change of signs.

97. Since $\sqrt{x^{2m}}=x^m$, it follows that

$$\sqrt{x^4}=x^2,\ \sqrt{x^6}=x^3,\ \sqrt{x^8}=x^4;$$

where it will be observed the index of the root is one-half the index of the given power.

98. The square root of a polynomial can generally be found by the following rule.

(i.) *Arrange the given quantity according to ascending or descending powers of some letter.*

(ii.) *The first term of the root is the square root of the leading term of the given quantity, from which its square is subtracted, leaving the first difference.*

(iii.) *The first divisor is twice the first term of the root added*

to the second term. The second term is the quotient of the leading term of the first difference divided by the leading term of the first divisor. The product of the first divisor and the second term of the root is subtracted from the first difference, leaving the second difference.

(iv.) *The second divisor is twice the sum of the first and second terms of the root added to the third term. The third term is the quotient of the leading term of the second difference divided by the leading term of the second divisor. The product of the second divisor and the third term of the root is then subtracted from the second difference, leaving the third difference.*

The process is thus continued until the difference is zero.

Examples.

(1.) Find the square root of $9x^2-12x+4$.

$$\begin{array}{r|l}
 & 9x^2-12x+4\,(3x-2 \\
 & 9x^2 \\
\hline
6x-2) & -12x+4 \\
 & -12x+4 \\
\hline
\end{array}$$

Here the first term $3x$ is the square root of the leading term of the given quantity, from which its square $9x^2$ is subtracted, leaving $-12x+4$. The leading term of the first divisor is $2\times 3x=6x$. This is divided into $-12x$, the leading term of the first difference, giving -2, the second term of the root, which is also the second term of the first divisor. The product of $6x-2$ and -2 is subtracted from the first difference, leaving remainder zero. The root is therefore $3x-2$.

(2.) Find the square root of $4x^4-12x^3+5x^2+6x+1$.

$$\begin{array}{r|l}
 & 4x^4-12x^3+5x^2+6x+1\,(2x^2-3x-1 \\
 & 4x^4 \\
\hline
4x^2-3x) & -12x^3+5x^2+6x+1 \\
 & -12x^3+9x^2 \\
\hline
4x^2-6x-1) & -4x^2+6x+1 \\
 & -4x^2+6x+1 \\
\hline
\end{array}$$

The first two terms, $2x^2$ and $-3x$, are found as in Ex. 1. The first two terms of the second divisor $= 2(2x^2 - 3x) = 4x^2 - 6x$, the leading term of which is divided into $-4x^2$, the leading term of the second difference, giving -1, the third term of the root, which is also the third term of the second divisor. The product of $4x^2 - 6x - 1$ and -1 is then subtracted from the second difference, leaving remainder zero.

The root is therefore $2x^2 - 3x - 1$.

The latter terms of the differences need not be expressed until the corresponding terms in the partial products are to be subtracted; thus in the foregoing example $6x + 1$ might have been omitted from the first difference.

(3.) Find the square root of $x^4 - 4x^3y + 10x^2y^2 - 12xy^3 + 9y^4$.

$$\begin{array}{rl}
 & x^4 - 4x^3y + 10x^2y^2 - 12xy^3 + 9y^4 \;(x^2 - 2xy + 3y^2 \\
 & x^4 \\
\hline
2x^2 - 2xy) & -4x^3y + 10x^2y^2 \\
 & -4x^3y + \;\;4x^2y^2 \\
\hline
2x^2 - 4xy + 3y^2) & \;\;6x^2y^2 - 12xy^3 + 9y^4 \\
 & \;\;6x^2y^2 - 12xy^3 + 9y^4 \\
\hline
\end{array}$$

In this example the given quantity is arranged according to descending powers of x, and the first two terms only of the first difference are expressed.

99. The reason for the rule given in the preceding Article will appear from the following method of considering the last example.

The given quantity is there seen to be equal to

$$\begin{array}{l}
x^4 \\
\quad -4x^3y + 4x^2y^2 \\
\qquad\qquad +6x^2y^2 - 12xy^3 + 9y^4
\end{array}$$

that is, to

$$\begin{array}{l}
(x^2)^2 \\
+(2x^2 - 2xy)(-2xy) \\
+(2x^2 - 4xy + 3y^2)\, 3y^2
\end{array}$$

which by Art. 89 is equal to $(x^2 - 2xy + 3y^2)^2$.

Now, since $x^4 = (x^2)^2$, the first term, x^2, of the root is the square root of x^4, the leading term of the given quantity. Also since $-4x^3y = 2x^2(-2xy)$, it follows that the second term $-2xy$ is the quotient of

$-4x^3y$, the leading term of the first difference, divided by $2x^2$, the leading term of the first divisor. Again, since $6x^2y^2 = 2x^2(3y^2)$, it follows that the third term $3y^2$ is the quotient of $6x^2y^2$, the leading term of the second difference, divided by $2x^2$, the leading term of the second divisor.

100. When the process for extracting the square root is applied to a quantity which is not an exact square, a result is obtained the square of which added to the last difference is equal to the given quantity.

Example.

Find three terms of the square root of $1-2x$.

$$\begin{array}{r|l} & 1-2x\ \left(1-x-\frac{x^2}{2}\right. \\ & 1 \\ \hline 2-x) & -2x \\ & -2x+x^2 \\ \hline 2-2x-\frac{x^2}{2}) & -x^2 \\ & -x^2+x^3+\frac{x^4}{4} \\ \hline & -x^3-\frac{x^4}{4} \end{array}$$

The square root is, therefore, $1-x-\frac{x^2}{2}$ and remainder $-x^3-\frac{x^4}{4}$. Hence $\left(1-x-\frac{x^2}{2}\right)^2 - x^3 - \frac{x^4}{4} = 1-2x$.

Exercise XXXII.

Find the square roots of

(1.) $4a^4b^2$, $25x^2y^6$, $81x^4y^8$.

(2.) $16x^2+40x+25$.

(3.) $36x^2-36x+9$.

(4.) $1+6x+9x^2$.

(5.) $x^2+\frac{x}{2}+\frac{1}{16}$.

(6.) $x^2-7x+\frac{49}{4}$.

(7.) $4x^2-\frac{1}{3}x+\frac{1}{144}$. (8.) $4x^2-12xy+9y^2$.

(9.) $x^4+4x^3+6x^2+4x+1$. (10.) $x^4+2x^3+3x^2+2x+1$.

(11.) $x^4-4x^3y+6x^2y^2-4xy^3+y^4$.

(12.) $4x^6-4x^5-11x^4+14x^3+5x^2-12x+4$.

(13.) Extract to three terms the square root of $1+x$.

101. The method of extracting the square root of a numerical quantity is founded on the Algebraical process, as will appear by comparing the examples given below. We shall first show how the number of figures in the root is determined by dividing the given quantity into periods.

Since $\sqrt{1}=1$, $\sqrt{100}=10$, $\sqrt{10000}=100$, $\sqrt{1000000}=1000$, &c., it follows that the square root of a number between 1 and 100, that is, containing 1 or 2 figures, lies between 1 and 10, and therefore contains 1 figure; the square root of a number between 100 and 10000, that is containing 3 or 4 figures, lies between 10 and 100, and therefore contains 2 figures; so likewise the square root of a number containing 5 or 6 figures contains 3 figures; and so on. If, therefore, we divide a number into periods of 2 figures each, beginning at the units, the number of such periods, whether complete or not, will be the number of figures in the root.

In Arithmetic if the root is $a+b+c+$ &c.,

$2a$ is called the *first trial-divisor*,
$2a+2b$ „ *second* „
$2a+2b+2c$ „ *third* „

and so on.

Instead of obtaining the divisors as in the previous examples, we may form them as below, where it will be observed that the sum of a divisor and its last term or digit gives the next trial-divisor.

Examples.

$$
\begin{array}{lll}
\text{(1.)}\ a & & a^2+2ab+b^2+2ac+2bc+c^2\ (a+b+c \\
\phantom{\text{(1.)}}\ \underline{a} & & \underline{a^2} \\
\phantom{\text{(1.)}}\ 2a+b &) & 2ab+b^2+2ac+2bc+c^2 \\
\phantom{\text{(1.)}}\ \underline{b} & & \underline{2ab+b^2} \\
\phantom{\text{(1.)}}\ 2a+2b+c &) & 2ac+2bc+c^2 \\
 & & \underline{2ac+2bc+c^2}
\end{array}
$$

300	10′69′29 (300+20+7
300	9 00 00
600+20	) 1 69 29
20	1 24 00
600+40+7	) 45 29
	45 29

In the numerical example, since the given number contains 3 periods, the root will contain 3 figures. The leading figure of the root, which is also the number of hundreds, will be 3, since the given number lies between $90000=300^2$ and $160000=400^2$. The second figure of the root, which is also the number of tens, is obtained by dividing the first remainder 16929 by the first trial-divisor 600. The third figure of the root, which is the number of units, is obtained by dividing the second remainder 4529 by the second trial-divisor 640.

Omitting all unnecessary figures, we may arrange and describe the work as follows:—

3	10′69′29 (327
3	9
62	)169
2	124
647	) 4529
	4529

The leading figure of the root is 3, the square of which is the greatest square number under 10, the first period; the square of 3 is subtracted from the first period, and to the remainder is annexed the second period 69 to form the first dividend 169. The first figure of the root is doubled to give the first trial-divisor 6, the division of which into 16 indicates the second figure of the root. The second figure of the root is annexed to the trial-divisor to form the first divisor 62, which is multiplied by the second figure of the root, and the product is subtracted from 169. To the remainder is annexed the third period to form the second dividend 4529. Under the divisor 62 is written its last digit, and the sum forms the second trial-divisor 64. The third and last figure of the root is 7, because when annexed to the trial-divisor to form the

second divisor, the product of 647 and 7 is equal to 4529, the last dividend.

(2.)	2	5′83′51′23′36 (24156
	2	4
	44	183
	4	176
	481	751
	1	481
	4825	27023
	5	24125
	48306	289836
		289836

CHAPTER XII.

THE HIGHEST COMMON MEASURE.

102. A QUANTITY is said to be of so many *dimensions*, in any letter, as are indicated by the index of the highest power of that letter involved in it.

Thus $3x^2-2x+4$ is of 2 dimensions in x; $3y^4+2y-5$ is of 4 dimensions in y; and ax^3-bx^2+c is of 3 dimensions in x.

103. A *whole expression*, or *quantity*, is one which involves no fractional forms.

Thus $3x^2$, $-5xy$, $2x^3-3x+4$, are whole expressions, as are also all positive and negative integers.

104. When two or more whole expressions are multiplied together, each is said to be a *measure* of the product, and the product is said to be a *multiple* of each factor.

Thus 1, 4, a, and b are measures of $4ab$; 1, 3, x and $x+1$ are measures of $3x^2+3x$; 5, x^2, $x-1$, and y^2-1 are measures of $5x^2(x-1)(y^2-1)$.

105. It must be carefully observed that the terms measure and multiple are to be used only in connection with whole expressions. In order, therefore, to obtain a multiple of a quantity it must be multiplied by another whole quantity; and to obtain a measure of a quantity it must be divided by a whole quantity, the quotient also being a whole quantity.

Thus the terms measure and multiple cannot be used in connection with $\frac{2}{3}ax^2$, $x^2-\frac{x}{2}+3$, because they involve fractions; whilst 1, 3, a, x, x^2, and $x-1$ are measures of $3ax^2(x-1)$,

because the quotient of the latter divided by each of the former is a whole quantity.

106. The *highest measure* of a quantity is the quantity divided by $+1$ or -1, that divisor being taken which will make the first term of the quotient positive.

Thus the highest measure of $-4x^2$ is $4x^2$, of $5x-7$ is $5x-7$, and of $-2x^2+x-3$ is $2x^2-x+3$.

107. The *lowest multiple* of a quantity is the quantity multiplied by $+1$ or -1, that multiplier being taken which will make the first term of the product positive.

Thus the lowest multiple of $2x-3$ is $2x-3$, and of $-x^3+x-5$ is x^3-x+5.

108. When one quantity is a measure of two or more others, it is said to be a *common measure* of those quantities.

Thus $2x$ is a common measure of $4x^2$ and $2x^2-4x$, and $x-1$ is a common measure of $2x-2$, x^2-2x+1, and x^2-1.

109. The *highest common measure* of two or more quantities is the common measure of highest dimensions and greatest numerical coefficient or coefficients.

Thus the common measures of $4x^3$ and $6x^2y$ are

$$1,\ 2,\ x,\ 2x,\ 2x^2,$$

of which the last is called the highest common measure (H.C.M.); the common measures of $4(x^2-1)$ and $6(x-1)^2$ are

$$1,\ 2,\ x-1,\ 2(x-1),$$

of which the last is the H.C.M.

110. We shall consider the process for finding the H.C.M. in the three following cases, namely—I. When one of the quantities is a mononomial. II. When the two quantities are polynomials, neither of which has a mononomial measure. III. When the two quantities are polynomials, one or both of which have mononomial measures.

111. I. (i.) When the given quantities consist of two or more mononomials, *their* H.C.M. *is the product of the* G.C.M.

of the numerical coefficients and the highest power or powers common to the several given quantities.

Examples.

(1.) Find the H.C.M. of $18ab^2x^3$ and $15a^3b^2$.

Here the G.C.M. of 18 and 15 is 3;

the highest power of a common to both is a;

" " b " " b^2;

and there is no power of x common to both;

$\therefore$ H.C.M. is $3ab^2$.

(2.) Find the H.C.M. of $12x^2y^3z^4$, $16x^3y^2z^3$, and $28x^4yz^5$.

The G.C.M. of 12, 16, and 28 is 4;

the highest power of x common to the three quantities is x^2;

" " y " " y;

" " z " " z^3;

$\therefore$ H.C.M. is $4x^2yz^3$.

112. (ii.) The H.C.M. of a mononomial and a polynomial is the H.C.M. of the mononomial and the H.C.M. of the several terms of the polynomial; and may be found by the following rule:—

Express the polynomial as a product one factor of which is the H.C.M. *of its several terms: the* H.C.M. *of this simple factor and the given mononomial will be the* H.C.M. *required.*

Examples.

(1.) Find the H.C.M. of $6ab^3$ and $8a^2b^3-12a^3b^2$.

Here $8a^2b^3-12a^3b^2=4a^2b^2\,(2b-3a)$, where $4a^2b^2$ is the H.C.M. of $8a^2b^3$ and $12a^3b^2$; and the H.C.M. of $4a^2b^2$ and $6ab^3$ is $2ab^2$, the H.C.M. required.

(2.) Find the H.C.M. of $15xy^2z^3$ and $10x^3y^2z^2-15x^2y^3z^2+20x^2y^2z^3$.

Here $10x^3y^2z^2-15x^2y^3z^2+20x^2y^2z^3 = 5x^2y^2z^2(2x-3y+4z)$, where $5x^2y^2z^2$ is the H.C.M. of $10x^3y^2z^2$, $15x^2y^3z^2$, and $20x^2y^2z^3$;

and the H.C.M. of $5x^2y^2z^2$ and $15xy^2z^3$ is $5xy^2z^2$, the H.C.M. required.

EXERCISE XXXIII.

Find the H.C.M. of

(1.) $12ab^2$ and $16a^2b^2$.
(2.) $15a^2b$ and $20ab^2$.
(3.) $9axy$ and $36xyz$.
(4.) $9ax^2y^2$ and $15a^2xz$.
(5.) $42a^2x^4y$ and $35a^3x^2y^4$.
(6.) ab^2cu^2v and $3a^2buv^2w$.
(7.) $8a^4b$, $12a^3b^2$, and $16a^2b^3$.
(8.) $30ax^4y^2$, $42bx^3y^3$, and $18cx^2y^4$.
(9.) $4ab^2$ and $12a^2bx-8ab^2y$.
(10.) $10ab^2c$ and $30a^3b^3+45a^2b^4$.
(11.) $10ab^2xy$ and $42ab^3cx-70b^4cy$.
(12.) $8uv^2w$ and $12u^3w^3-24u^2vw^3+36u^2w^4$.

113. II. When two polynomials, neither of which contains a mononomial measure (other than unity), involve powers of a single letter, their H.C.M. can be obtained by the following rule:—

(i.) *Having arranged the given quantities according to descending powers, choose that one which is not of lower dimensions than the other as divisor.*

(ii.) *Divide this into the other multiplied by the least number which will make its leading term a multiple of the leading term of the divisor. When this number is unity, actual multiplication may be dispensed with.*

(iii.) *Divide the first difference by the highest mononomial measure contained in it. When this measure is* $+1$, *actual division may be dispensed with.*

(iv.) *Repeat the steps* (i.), (ii.), (iii.), *with respect to this last quotient (or difference) and the first divisor; and so on, until there is no difference.*

The last divisor will be the H.C.M. *required.*

It will be observed that no fractions occur in the process, and that the leading signs of all divisors are made positive.

Examples.

(1.) Find the H.C.M. of $2x^2-7x+5$ and $3x^2-7x+4$.

$$\begin{array}{r|l}
 & 3x^2-7x+4 \\
 & 2 \\
\hline
2x^2-7x+5) & 6x^2-14x+8\ (3 \\
 & 6x^2-21x+15 \\
\hline
 & 7)\ 7x-7 \\
\hline
 & x-1
\end{array}$$

Here since the dimensions of the two given quantities are equal, either one may be made the divisor. $2x^2-7x+5$ being taken as divisor, $3x^2-7x+4$ is multiplied by 2 in order to make the leading term $6x^2$ a multiple of the leading term $2x^2$ of the divisor. The first difference $7x-7$ is divided by its highest mononomial measure 7.

In the next step $x-1$ and $2x^2-7x+5$ are to be treated in the same manner as the given quantities.

$$\begin{array}{r|l}
x-1) & 2x^2-7x+5\ (2x \\
 & 2x^2-2x \\
\hline
 & -5)\ -5x+5 \\
\hline
 & x-1
\end{array}$$

The leading term of $2x^2-7x+5$ is a multiple of the leading term of $x-1$, and therefore the multiplication of the former by 1 is omitted. The difference $-5x+5$ is divided by its highest mononomial measure -5.

In the next step the quotient $x-1$ and divisor $x-1$ are to be treated as the previous quotient and divisor were.

$$\begin{array}{r|l}
x-1) & x-1\ (1 \\
 & x-1 \\
\hline
\end{array}$$

The process thus terminates and the H.C.M. is the last divisor $x-1$.

Whenever as in the second step the difference is a multiple of the divisor, the division may be continued and the work of the last step avoided. Thus

$$
\begin{array}{r|l}
x-1) & 2x^2-7x+5\ (2x-5 \\
& \underline{2x^2-2x} \\
& -5x+5 \\
& \underline{-5x+5}
\end{array}
$$

The whole work may be arranged as follows:

$$
\begin{array}{l}
3x^2-7x+\ 4 \\
\underline{2} \\
2x^2-7x+5)\ 6x^2-14x+\ 8\ (3 \\
\underline{6x^2-21x+15} \\
7)\ 7x-\ 7 \\
x-\ 1)\ 2x^2-7x+5\ (2x-5 \\
\underline{2x^2-2x} \\
-5x+5 \\
\text{H.C.M.}=x-1.\underline{-5x+5}
\end{array}
$$

(2.) Find the H.C.M. of x^2+2x-3 and x^2+5x+6.

$$
\begin{array}{l}
x^2+2x-3)\ x^2+5x+6\ (1 \\
\underline{x^2+2x-3} \\
3)\ 3x+9 \\
x+3)\ x^2+2x-3\ (x-1 \\
\underline{x^2+3x} \\
-x-3 \\
\text{H.C.M.}=x+3.\underline{-x-3}
\end{array}
$$

Here the multiplication of x^2+5x+6 by unity is unnecessary. The other steps are similar to Ex. 1.

(3.) Find the H.C.M. of $2x^3-7x-2$ and $2x^2-x-6$.

$$
\begin{array}{rl}
2x^2-x-6) & 2x^3-7x-2\,(x,\,1 \\
& \underline{2x^3-x^2-6x} \\
& \qquad x^2-x-2 \\
& \qquad\qquad\quad 2 \\
& \overline{\quad 2x^2-2x-4} \\
& \underline{\quad 2x^2-\;x-6} \\
& \underline{\;-1)-x+2\;} \\
& \qquad\qquad x-2)\;2x^2-x-6\,(2x+3 \\
& \qquad\qquad\qquad\;\; \underline{2x^2-4x} \\
& \qquad\qquad\qquad\qquad 3x-6 \\
\text{H.C.M.}=x-2. & \qquad\qquad\qquad\qquad \underline{3x-6}
\end{array}
$$

Here $2x^2-x-6$ is used as divisor in the second step, the dimensions of the first difference being 2. The partial quotients x, 1 of $2x^3-7x-2$ and $2x^2-2x-4$ divided by $2x^2-x-6$ are separated by a comma to distinguish them from parts of an ordinary quotient.

(4.) Find the H.C.M. of

$$4x^2+3x-10 \text{ and } 4x^3+7x^2-3x-15.$$

$$
\begin{array}{rl}
4x^2+3x-10) & 4x^3+7x^2-\;3x-15\,(x+1 \\
& \underline{4x^3+3x^2-10x} \\
& \qquad 4x^2+7x-15 \\
& \qquad \underline{4x^2+3x-10} \\
& \qquad\qquad 4x-5)\;4x^2+3x-10\,(x+2 \\
& \qquad\qquad\qquad\quad \underline{4x^2-5x} \\
& \qquad\qquad\qquad\qquad\quad 8x-10 \\
\text{H.C.M.}=4x-5. & \qquad\qquad\qquad\qquad\quad \underline{8x-10}
\end{array}
$$

In this example there is no necessity to introduce or suppress any mononomial factors.

114. The process of the foregoing examples will frequently enable us to find the H.C.M. of polynomials involving powers of several letters, as in the following

Example.

Find the H.C.M. of $2x^2+xy-3y^2$ and $3x^2-4xy+y^2$.

$$
\begin{array}{rl}
 & 3x^2-4xy+y^2 \\
 & \qquad\qquad\quad 2 \\
\hline
2x^2+xy-3y^2) & 6x^2-8xy+2y^2\ (3 \\
 & \underline{6x^2+3xy-9y^2} \\
 & -11y)\ -11xy+11y^2 \\
 & \qquad x-y)\ 2x^2+xy-3y^2\ (2x+3y \\
 & \qquad\qquad\ \underline{2x^2-2xy} \\
 & \qquad\qquad\qquad 3xy-3y^2 \\
\text{H.C.M.}=x-y. & \qquad\qquad\qquad \underline{3xy-3y^2}
\end{array}
$$

Here the mononomial factor $-11y$ is suppressed.

115. The reason for the rule in Art. 113 will appear from the following proposition and its application in the next Art.

When one quantity is a measure of two others, it will measure the sum and difference of any multiples of them.

Let the quantities be A, B, C; and let A measure B and C, so that $B=mA$, $C=nA$, where m and n are whole quantities.

Take any multiples pB, qC of B, C, where p and q are any whole quantities whatsoever. Then, since $pB=pmA$, $qC=qnA$,

$$pB\pm qC=pmA\pm qnA=(pm\pm qn)A.$$

$$\therefore\ \frac{pB\pm qC}{A}=pm\pm qn;$$

that is, A is a measure of $pB\pm qC$, the sum or difference of any multiples of B and C, because the quotient $pm\pm qn$ is a whole quantity.

Thus, $2x^2$, which is a measure of $6x^3$ and $8x^2y$, will measure $6x^3(-2a)-8x^2y(-3x)$, $6x^3+8x^2y$, $6x^3-8x^2y(4xy)$, &c.

116. Suppose, now, that A and B denote two polynomials (as in Art. 113), neither of which contains a mononomial measure other than unity; and let the dimensions of A be not greater than the dimensions of B. Divide A into B multiplied by a mononomial whole quantity a, which makes its first term a multiple of the first term of A; and

divide the difference C by the highest mononomial measure which it contains, and let the quotient be D.

$$\begin{array}{r|l} & B \\ & a \\ \hline A) & aB\,(b \\ & bA \\ \hline c) & C \\ \hline & D \end{array}$$

Now, C being equal to $aB-bA$, or the difference of two multiples of A and B, is a multiple of all the common measures of A and B, and therefore of their H.C.M.

Again, every common measure of C and A is a measure of $C+bA$, or aB, and therefore of B, because A has no mononomial measure.

Hence the H.C.M. of A and B is the same as the H.C.M. of A and C, which is the same as the H.C.M. of A and D, because A has no mononomial measure.

The problem is thus reduced to finding the H.C.M. of A and D.

These two quantities, A and D, are then treated in precisely the same manner as A and B; and the process is continued until it terminates as follows, when the last divisor, P (suppose), is a measure of the last dividend Q.

$$\begin{array}{r|l} P) & Q\,(r \\ & rP \\ \hline & 0 \end{array}$$

The problem is thus finally reduced to finding the H.C.M. of P and Q. This is evidently P.

Hence the last divisor in the above process will be the H.C.M. required.

Exercise XXXIV.

Find the H.C.M. of

(1.) $3x^2+2x-21$ and $5x^2+13x-6$.

(2.) $2x^2+x-3$ and $3x^2-4x+1$.

(3.) x^2-5x+6 and x^2-6x+9.

(4.) $x^2+10x+21$ and $x^2-2x-15$.

(5.) $2x^2+x-15$ and $2x^2-19x+35$.

(6.) x^2-4x+3 and $4x^3-9x^2-15x+18$.

(7.) $x^2+10x+25$ and $x^3+15x^2+75x+125$.

(8.) $x^3-6x^2+11x-6$ and $x^3-x^2-14x+24$.

(9.) $x^3-3x^2-9x+27$ and $3x^3-x^2-27x+9$.

(10.) $3x^2-22x+32$ and $x^3-11x^2+32x-28$.

(11.) $7x^2-12x+5$ and $2x^3+x^2-8x+5$.

(12.) $5x^3+2x^2-15x-6$ and $7x^3-4x^2-21x+12$.

(13.) $2x^3+9x^2+4x-15$ and $4x^3+8x^2+3x+20$.

(14.) $x^3-6x^2+11x-6$ and $x^4-2x^3-13x^2+14x+24$.

(15.) x^4-2x^2+1 and $x^4-4x^3+6x^2-4x+1$.

(16.) $x^2+xy-12y^2$ and $x^2-5xy+6y^2$.

(17.) $2x^2+3xy+y^2$ and $3x^2+2xy-y^2$.

(18.) $x^3+x^2y+xy+y^2$ and x^4-y^2.

(19.) $5x^2+26xy+33y^2$ and $7x^2+19xy-6y^2$.

(20.) $3x^4-x^2y^2-2y^4$ and $2x^4+3x^3y-2x^2y^2-3xy^3$.

117. III. The H.C.M. of two polynomials involving mononomial measures is found as follows:

Express each polynomial as the product of a mononomial and a polynomial which contains no mononomial measure. Then the H.C.M. *required is the product of the* H.C.M. *of the mononomial factors and the* H.C.M. *of the polynomial factors.*

Example.

Find the H.C.M. of

$$8x^4y+12x^3y+4x^2y \text{ and } 6x^3y^2-6x^2y^2-12xy^2.$$

Here the given quantities are equal to

$$4x^2y(2x^2+3x+1) \text{ and } 6xy^2(x^2-x-2).$$

The H.C.M. of $4x^2y$ and $6xy^2$ is $2xy$; and the H.C.M. of

$$2x^2+3x+1 \text{ and } x^2-x-2 \text{ is } x+1.$$

$\therefore$ the H.C.M. required is $2xy(x+1)$.

118. The H.C.M. of three polynomials is the H.C.M. of any one and of the H.C.M. of the other two.

Example.

Find the H.C.M. of x^4-1, x^4+2x^2-3, and $2x^4+2x^3+3x+3$.

The H.C.M. of x^4-1 and x^4+2x^2-3 is x^2-1; and the H.C.M. of x^2-1 and $2x^4+2x^3+3x+3$ is $x+1$, the H.C.M. required.

Exercise XXXV.

Find the H.C.M. of

(1.) $12ax^4-27ax^2$ and $2a^2x^3+a^2x^2-3a^2x$.

(2.) $10x^2+40x+30$ and $4x^3-16x^2-84x$.

(3.) $2x^5-6x^3-4x^2$ and $3x^4-3x^3-12x$.

(4.) $2x^2+x-3$, x^2-1, and x^2+4x-5.

(5.) $6x^2-x-2$, $21x^2-17x+2$, and $15x^2+5x-10$.

119. When all the component factors of the given quantities are known or can be determined, the H.C.M. may be found by the rule of Art. 111.

Examples.

(1.) Find the H.C.M. of

$$4(x-1)^2(x+2)^3 \text{ and } 6(x-1)^3(x+2).$$

The G.C.M. of 4 and 6 is 2;

the highest power of $x-1$ common to both is $(x-1)^2$;

" " $x+2$ " " $x+2$.

$\therefore$ the H.C.M. required is $2(x-1)^2(x+2)$.

(2.) Find the H.C.M. of

$$8a^2x(x^3-1),\ 12ax^2(x^4-1), \text{ and } 20ax(x^2-1).$$

By Art. 80, $\quad 8a^2x(x^3-1)=8a^2x(x-1)(x^2+x+1)$;

by Art. 79, $\quad 12ax^2(x^4-1)=12ax^2(x^2-1)(x^2+1)$

$$=12ax^2(x-1)(x+1)(x^2+1);$$

$$20ax(x^2-1)=20ax(x+1)(x-1).$$

Now the G.C.M. of 8, 12, and 20 is 4;
the highest power of a common to the three quantities is a;
" " x " " " x;
" " $x-1$ " " " $x-1$;
and the other factors $x+1$, x^2+1, x^2+x+1 are not common.

$\therefore$ the H.C.M. required is $4ax(x-1)$.

EXERCISE XXXVI.

Find the H.C.M. of

(1.) $2a(x-a)^2$ and $3ax(x+a)(x-a)$.

(2.) $6a^2(x+2)(x-3)$ and $8ax(x-3)(x+3)$.

(3.) $ax^2-2ax+a$ and $2a^2x^2-2a^2$.

(4.) x^2-1, x^3+1, and x^4-1.

(5.) $x+2$, x^2-4, and x^3+8.

(6.) $3x^3-81$, x^2-6x+9, and $2x^3-18x$.

CHAPTER XIII.

THE LOWEST COMMON MULTIPLE.

120. WHEN one quantity is a multiple of two or more others, it is said to be a *common multiple* of those quantities.

Thus $12x^2$ is a common multiple of $2x$ and $3x^2$; and $15x(x-1)$ of 3, 5, $15x$ and $x-1$.

121. The *lowest common multiple* (L.C.M.) of two or more quantities is the common multiple of lowest dimensions and least numerical coefficient or coefficients.

Thus of the following common multiples of $2x$ and $3x^2$, namely,

$$\begin{gathered} 6x^2,\ 12x^2,\ 18x^2, \\ 6x^3,\ 12x^3,\ 18x^3, \\ 6x^4,\ 12x^4,\ 18x^4,\ \&c., \end{gathered}$$

the first $6x^2$ is called the lowest.

122. The L.C.M. of two quantities is found by the following rule:—

(i.) *If they contain no common measure except unity, their* L.C.M. *is their product.*

Thus, the L.C.M. of $4x$ and $7ab$ is $28abx$.

(ii.) *If they contain a common measure, their* L.C.M. *is equal to one of the given quantities multiplied by the quotient of the other divided by their* H.C.M.

It will be generally found most convenient to express the L.C.M. as the product of several factors.

Examples.

(1.) Find the L.C.M. of $6x^2y$ and $9xy^2$.

The H.C.M. of these quantities is $3xy$.

$\therefore$ by the rule, L.C.M. $=\dfrac{6x^2y}{3xy}\times 9xy^2=18x^2y^2$.

(2.) Find the L.C.M. of $2x^2-7x+5$ and $3x^2-7x+4$.

The H.C.M. is found to be $x-1$; and since $\dfrac{2x^2-7x+5}{x-1}=2x-5$, the L.C.M. will be $(2x-5)(3x^2-7x+4)$.

123. The following is the proof of the rule given in the preceding Article:—

Let the two quantities be denoted by A and B, and their H.C.M. by C; and let $A=aC$, $B=bC$, where a and b are whole expressions which have no common measure except unity.

Then, since the L.C.M. of a and b is ab, the L.C.M. of aC $(=A)$ and bC $(=B)$ is $abC=\dfrac{aC.\,bC}{C}=\dfrac{AB}{C}=\dfrac{A}{C}.B=\dfrac{B}{C}.A$.

124. *The* L.C.M. *of three quantities is the* L.C.M. *of any one and the* L.C.M. *of the remaining two.*

The L.C.M. *of four quantities is the* L.C.M. *of any one and the* L.C.M. *of the remaining three.*

And so on.

Example.

Find the L.C.M. of $3x^2y^2z$, $6xyz^2$, and $10x^2yz^3$.

The L.C.M. of $3x^2y^2z$ and $6xyz^2$ is $6x^2y^2z^2$; and the L.C.M. of $6x^2y^2z^2$ and $10x^2yz^3$ is $30x^2y^2z^3$.

125. When all the component factors of the given quantities are known or can be found, their L.C.M. may also be obtained *by multiplying the* L.C.M. *of the numerical factors by the highest power or powers of the several factors that occur in the given quantities.*

Examples.

(1.) Find the L.C.M. of $6x^2yz^2$, $4x^2y^3z$, and $8x^4y^2z$.

The L.C.M. of 6, 4, and 8 is 24; the highest power of x which occurs amongst the factors of the given quantities is x^4; and the highest powers of y and z are, respectively, y^3 and z^2.

Therefore the L.C.M. required is $24x^4y^3z^2$.

(2.) Find the L.C.M. of $15ab(a-b)$, $21a(a+b)(a-b)$, and $35b^2(a+b)$.

The L.C.M. of 15, 21, and 35 is 105; the highest powers of $a, b, a-b, a+b$, which occur amongst the given quantities, are, respectively, a, b^2, $a-b$, $a+b$.

Therefore the L.C.M. required is $105ab^2(a-b)(a+b)$.

(3.) Find the L.C.M. of x^2-1, x^3-1, and x^3+1.

Here
$$x^2-1=(x+1)(x-1);$$
$$x^3-1=(x-1)(x^2+x+1);$$
$$x^3+1=(x+1)(x^2-x+1).$$

$$\therefore \text{ the L.C.M.}=(x+1)(x-1)(x^2+x+1)(x^2-x+1)$$
$$=(x^2-1)(x^4+x^2+1).$$

EXERCISE XXXVII.

Find the L.C.M. of

(1.) $3abx$, $2bxy$. (2.) $8a^2xy$, $12ax^2y$. (3.) ab^2, bc^2, ca^2.

(4.) $8a^2bc$, $12ab^2c$, $24abc^2$.

(5.) $12bcu^2$, $16cav^2$, $20abw^2$, $40a^2b^2c^2$.

(6.) $x^2-7x+12$, x^2-x-6. (7.) $2x^2-5x-3$, $4x^2+4x+1$.

(8.) $3x^2-11x+6$, $2x^2-7x+3$.

(9.) $x^3-4ax^2+5a^2x-2a^3$, $x^3-2a^2x-4a^3$.

(10.) $8(x^2-1)$, $12(x-1)^2$.

(11.) x^2-1, $(x-1)^2$, $(x+1)^2$.

(12.) $x^2(x+y)$, $xy(x-y)$, $y^2(x+y)$.

(13.) a^4+a^3b, $ab-b^2$, a^2-b^2.

(14.) $2x(x^2+x+1)$, $3x^3-3$, $4x^2-4$.

(15.) p^2+q^2, p^2-q^2, p^3+q^3.

(16.) p^3-1, p^4-1, p^6-1.

(17.) $(a-b)(a-c)$, $(b-c)(b-a)$, $(c-a)(c-b)$.

(18.) $8a^2b(a-b)$, $12ab(b-a)$, $3(a^2-b^2)$, $b^2a^2(b^2-a^2)$.

CHAPTER XIV.

FRACTIONS.

126. When one quantity is not exactly divisible by another, the quotient is represented by writing them in the form of a fraction.

Thus, $\frac{-2}{+3}$ denotes the quotient of -2 divided by $+3$; $\frac{-2a}{-3x}$ the quotient of $-2a$ divided by $-3x$; $\frac{x-1}{x^2-3x+4}$ the quotient of $x-1$ divided by x^2-3x+4.

127. A fraction is not altered in value by multiplying or dividing the numerator and denominator by the same quantity.

Thus, $\frac{-2}{+3}=\frac{-2\times3}{+3\times3}=\frac{-6}{+9}$; $\frac{+6}{-8}=\frac{+6\times-4}{-8\times-4}=\frac{-24}{+32}$;

$$\frac{-20}{-25}=\frac{-20\div-5}{-25\div-5}=\frac{+4}{+5};\quad \frac{-4}{+5}=\frac{-4\times\frac{2}{3}}{+5\times\frac{2}{3}}=\frac{-\frac{8}{3}}{+\frac{10}{3}};$$

$$\frac{a}{b}=\frac{ac}{bc}=\frac{-ax}{-bx};$$

$$\frac{x-1}{2x-3}=\frac{(x-1)(x+1)}{(2x-3)(x+1)}=\frac{x^2-1}{2x^2-x-3}.$$

128. The statement in the preceding Article depends on the two following propositions:—

I. *The numerical value of a fraction is unaltered by multiplying or dividing its numerator and denominator by the same quantity.*

(i.) Let a, b, m, be integers. Then, since $\frac{a}{b}$ denotes a of the b parts into which a unit is divided, it follows that

$$\frac{a}{b} \times m = \frac{ma}{b} \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{a}{b} \div m = \frac{a}{mb} \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{a}{b} = \frac{ma}{mb} \quad . \quad . \quad . \quad . \quad (3)$$

Before considering the case where a, b, m are fractional, we shall show how the operations of multiplication and division of numerical fractions must be performed.

Let a, b, c, d, m be integers. Then, since $\frac{mc}{c} = m$, it follows that

$$\frac{a}{b} \cdot \frac{mc}{c} = \frac{a}{b} \times m = \frac{ma}{b}$$

But by (3), $\frac{ma}{b} = \frac{mac}{bc}$

$$\therefore \frac{a}{b} \cdot \frac{mc}{c} = \frac{mac}{bc} \quad . \quad . \quad . \quad (4)$$

Hence, if $\frac{a}{b}$ be multiplied by an integer *in a fractional form*, the product is a fraction whose numerator is the product of the numerators and denominator the product of the denominators. If, now, we wish to find the product of $\frac{a}{b}$ and $\frac{c}{d}$, where $\frac{c}{d}$ is not equal to an integer, the operation of multiplication must be in accordance with this rule; for any application of the term *multiplication* to cases where its primary meaning (which is *repetition*) does not apply, must not be inconsistent with the cases where it does apply.

$$\therefore \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Again, since division is the inverse of multiplication, both in its primary and extended applications, it follows from (5) that

$$\frac{ac}{bd} \div \frac{c}{d} = \frac{a}{b}$$

$$= \frac{acd}{bcd} \quad \text{by (3)}$$

$$= \frac{ac}{bd} \cdot \frac{d}{c} \quad \text{by (5)}$$

Also
$$\frac{ac}{bd} \div \frac{a}{b} = \frac{c}{d}$$
$$= \frac{abc}{abd} \quad \text{by (3)}$$
$$= \frac{ac}{bd} \cdot \frac{b}{a} \quad \text{by (5)}$$

Hence it follows that the quotient of the fraction $\frac{a}{b}$ divided by $\frac{c}{d}$ is equal to the product of $\frac{a}{b}$ and $\frac{d}{c}$; that is,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \quad . \quad . \quad (6)$$

(ii.) Now, let a, b, m be fractional, which will include the case where some of them may be integers.

Let
$$a = \frac{x}{y}, \; b = \frac{z}{u}, \; m = \frac{c}{d}.$$

Then
$$\frac{a}{b} = \frac{\frac{x}{y}}{\frac{z}{u}} = \frac{xu}{yz} \quad \text{by (6);}$$

And
$$\frac{ma}{mb} = \frac{\frac{c}{d} \cdot \frac{x}{y}}{\frac{c}{d} \cdot \frac{z}{u}} = \frac{\frac{cx}{dy}}{\frac{cz}{du}} \quad \text{by (5)}$$
$$= \frac{cxdu}{czdy} \quad \text{by (6)}$$
$$= \frac{xu}{yz} \quad \text{by (3)}$$
$$\therefore \frac{a}{b} = \frac{ma}{mb}.$$

II. If, in the fraction $\frac{a}{b}$, a and b denote positive and negative quantities, the sign of the fraction depends on the signs of the numerator and denominator. These signs will be either like or unlike, and on multiplying or dividing by a positive or negative quantity they will still be like or unlike, and the sign of the fraction will therefore remain unchanged.

For example, multiplying by -1,

$$\frac{+2}{+3}=\frac{-2}{-3}=+\frac{2}{3}, \text{ a positive fraction;}$$

$$\frac{+3}{-4}=\frac{-3}{+4}=-\frac{3}{4}, \text{ a negative fraction.}$$

129. A fraction which involves fractional coefficients in the terms of the numerator and denominator can always be reduced to one whose numerator and denominator are whole expressions by multiplying the numerator and denominator by the L.C.M. of the several denominators.

Thus, $\dfrac{x-\frac{1}{2}}{2x^2-\frac{3}{4}x+\frac{1}{6}}$ is reduced by multiplying numerator and denominator by 12, the L.C.M. of 2, 4, and 6, to the equivalent from $\dfrac{12x-6}{24x^2-9x+2}$.

130. A fraction is said to be in *lowest terms* when its numerator and denominator contain no common measure, except unity. Hence *a fraction is reduced to lowest terms by dividing its numerator and denominator by their* H.C.M.

Examples.

(1.) Reduce $\dfrac{8a^2b^3x}{12a^3b^2y}$ to lowest terms.

The H.C.M. of $8a^2b^3x$ and $12a^3b^2y$ is $4a^2b^2$.

$$\therefore \frac{8a^2b^3x}{12a^3b^2y}=\frac{2bx}{3ay}.$$

(2.) Reduce $\dfrac{x^2-1}{x^3+1}$ to lowest terms.

The H.C.M. $= x+1$.

$$\therefore \frac{x^2-1}{x^3+1}=\frac{x-1}{x^2-x+1}.$$

(3.) Reduce $\dfrac{2x^2+3x-2}{2x^2-5x+2}$ to lowest terms.

The H.C.M. $= 2x-1$.

$$\therefore \frac{2x^2+3x-2}{2x^2-5x+2}=\frac{x+2}{x-2}.$$

EXERCISE XXXVIII.

Reduce to lowest terms—

(1.) $\frac{15a^2x}{45ax^2}$. (2.) $\frac{abx}{acy}$. (3.) $\frac{6x^2yz^3}{8xy^2z^4}$. (4.) $\frac{5ab}{a^2b-ab^2}$.

(5.) $\frac{6a^2b^2}{9ab^3-18a^3b}$. (6.) $\frac{x+1}{x^2-1}$. (7.) $\frac{x-2}{x^2-4}$.

(8.) $\frac{x^3-1}{x^6-1}$. (9.) $\frac{x^2-y^2}{x^3-y^3}$. (10.) $\frac{x-3}{x^2-4x+3}$.

(11.) $\frac{x^2-7x+10}{x^2-x-2}$. (12.) $\frac{x^2+7x+12}{x^2-x-20}$.

(13.) $\frac{3+10x+3x^2}{3+8x-3x^2}$. (14.) $\frac{x^2-3x-70}{x^3-39x+70}$.

(15.) $\frac{x^3-6x-9}{x^4+3x^3-9x-9}$. (16.) $\frac{12x^2-15x+3}{6x^3-6x^2+2x-2}$.

(17.) $\frac{x^4-4x^2y^2}{x^3-6x^2y+12xy^2-8y^3}$. (18.) $\frac{x^3-3x^2y+3xy^2-y^3}{x^3-x^2y-xy^2+y^3}$.

131. Fractions are said to be *like* or *unlike*, according as they have the same or different denominators.

Thus $\frac{a}{x}$, $\frac{b}{x}$ are like fractions, as are also $\frac{3x}{x^2-1}$, $\frac{2a}{x^2-1}$; $\frac{2c}{3b}$, $\frac{2a}{3x}$ are unlike fractions.

132. Unlike fractions are reduced to like fractions *by multiplying the numerator and denominator of each fraction by the quotient of the* L.C.M. *of the several denominators divided by its denominator.*

The common denominator will accordingly be the L.C.M. of the several denominators.

Examples.

(1.) Reduce $\frac{2a}{3b}$, $\frac{3c}{4d}$ to like fractions.

The L.C.M. of $3b$ and $4d$ is $12bd$.

The multiplier for the first fraction is $\frac{12bd}{3b} = 4d$;

$$\therefore\ \frac{2a}{3b} = \frac{8ad}{12bd}.$$

The multiplier for the second fraction is $\frac{12bd}{4d} = 3b$;

$$\therefore\ \frac{3c}{4d} = \frac{9bc}{12bd}.$$

(2.) Reduce $\frac{1}{a-b}$, $\frac{2}{b-a}$ to like fractions.

The L.C.M. of $a-b$ and $b-a$ is $a-b$, the quotient of which divided by $b-a$ is -1;

$$\therefore\ \frac{2}{b-a} = \frac{-2}{a-b}.$$

(3.) Reduce $\frac{1}{x-1}$, $\frac{x-1}{x^2+x+1}$, $\frac{x-2x^2}{x^3-1}$ to like fractions.

The L.C.M. of the denominators is x^3-1, the quotients of which divided by $x-1$, x^2+x+1, x^3-1 are, respectively, x^2+x+1, $x-1$, 1.

$$\therefore\ \frac{1}{x-1} = \frac{x^2+x+1}{x^3-1},$$

$$\frac{x-1}{x^2+x+1} = \frac{x^2-2x+1}{x^3-1},$$

$$\frac{x-2x^2}{x^3-1} = \frac{x-2x^2}{x^3-1}.$$

Exercise XXXIX.

Reduce to like fractions:—

(1.) $\frac{1}{a}, \frac{2}{b}$. (2.) $\frac{1}{a^2}, \frac{1}{ab}$. (3.) $\frac{2}{x}, \frac{3}{xy}, \frac{4}{xyz}$.

(4.) $\frac{2}{a}, \frac{x}{-a}$. (5.) $\frac{1}{ax}, \frac{2}{-xy}, \frac{3}{axy}$. (6.) $\frac{a}{x-1}, \frac{2a}{x^2-1}$.

(7.) $\frac{1}{x+1}, \frac{2}{x+3}$. (8.) $\frac{2}{x-1}, \frac{3-x}{1-x}$. (9.) $\frac{a}{a-b}, \frac{1-a}{b-a}$.

(10.) $\frac{4}{x}, \frac{3}{x+1}, \frac{1}{x^2-1}$. (11.) $\frac{x-1}{x^2-x+1}, \frac{1}{x+1}, \frac{3x}{x^3+1}$.

(12.) $\frac{2}{(x+2)(x-1)}, \frac{3}{(1-x)(2-x)}, \frac{1}{(x-2)(x+2)}$.

(13.) $\frac{1}{(b-a)(c-a)}, \frac{1}{(a-b)(c-b)}, \frac{1}{(a-c)(b-c)}$.

(14.) $\frac{1}{a(a-b)(x-a)}, \frac{1}{b(b-a)(x-b)}, \frac{1}{abx}$.

Addition and Subtraction.

133. The operations of addition and subtraction of fractions may be denoted by the signs + and − respectively.

Thus, *the sum of* $\frac{a}{x-1}, \frac{-2b}{x^2-3}$, *and* $\frac{x-2}{3x-5}$, may be denoted by

$$\frac{a}{x-1}+\frac{-2b}{x^2-3}+\frac{x-2}{3x-5};$$

and *the difference between* $\frac{4x}{3x^2-1}$ *and* $\frac{x-1}{5x-4}$, by

$$\frac{4x}{3x^2-1}-\frac{x-1}{5x-4}.$$

Thus also $+\frac{a}{b}$ denotes the fraction $\frac{a}{b}$ *to be added to*, and $-\frac{a}{b}$ the fraction $\frac{a}{b}$ *to be subtracted from* some quantity not expressed.

134. Sums and differences of fractions when expressed as single fractions are said to be *simplified*, the operation being performed according to the three following rules:—

(i.) *The sum of any number of like fractions is a like fraction whose numerator is the sum of their numerators.*

Thus, $$\frac{4}{x-1}+\frac{-x}{x-1}+\frac{2x-3}{x-1}=\frac{4-x+2x-3}{x-1}=\frac{x+1}{x-1}.$$

(ii.) *The difference between two like fractions is a like fraction whose numerator is the difference between their numerators.*

Examples.

(1.) $$\frac{5x-4}{2x-3}-\frac{6}{2x-3}=\frac{5x-4-6}{2x-3}=\frac{5x-10}{2x-3}.$$

(2.) $$\frac{4x}{x^2-1}-\frac{-2x}{x^2-1}=\frac{4x+2x}{x^2-1}=\frac{6x}{x^2-1}.$$

It will be observed that the $-$ before the second fraction changes the sign of $-2x$.

(3.) $$\frac{3x^2-x+5}{x^3+1}-\frac{x^2-6x-8}{x^3+1}=\frac{3x^2-x+5-(x^2-6x-8)}{x^3+1}$$
$$=\frac{3x^2-x+5-x^2+6x+8}{x^3+1}$$
$$=\frac{2x^2+5x+13}{x^3+1}.$$

In this case the $-$ before the second fraction changes the signs of all the terms of x^2-6x-8.

It appears from the preceding rules that

$$-\frac{a-b}{c}=+\frac{-a+b}{c}$$

and conversely; in other words, *the subtraction of a fraction is equivalent to the addition of a like fraction, whose numerator is the numerator of the former with its sign or signs changed.*

Thus, $$-\frac{x^2-2x+3}{x^3-1}=+\frac{-x^2+2x-3}{x^3-1};$$

$$+\frac{-2x+5}{3x^2-7}=-\frac{2x-5}{3x^2-7}.$$

(iii.) Addition and subtraction of unlike fractions are performed *by reducing the unlike to like fractions and proceeding as above* (i.), (ii.).

Examples.

(1.) Simplify $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}$.

Since the L.C.M. of bc, ca, ab is abc, this sum is equal to

$$\frac{a}{abc}+\frac{b}{abc}+\frac{c}{abc}=\frac{a+b+c}{abc}.$$

(2.) Simplify $\frac{1}{1+a}+\frac{1}{1-a}-\frac{2a}{1-a^2}$.

The L.C.M. of the denominators is $1-a^2$.

$$\begin{aligned}\therefore \frac{1}{1+a}+\frac{1}{1-a}-\frac{2a}{1-a^2} &= \frac{1-a}{1-a^2}+\frac{1+a}{1-a^2}-\frac{2a}{1-a^2}\\ &= \frac{1-a+1+a-2a}{1-a^2}\\ &= \frac{2-2a}{1-a^2}\\ &= \frac{2}{1+a}.\end{aligned}$$

(3.) Simplify $\frac{1}{(x+y)^2}-\frac{1}{y^2-x^2}-\frac{1}{(x-y)^2}$.

The L.C.M. of the denominators is $(x+y)^2\,(x-y)^2 = (x^2-y^2)^2$.

$$\begin{aligned}\therefore\ &\frac{1}{(x+y)^2}-\frac{1}{y^2-x^2}-\frac{1}{(x-y)^2}\\ &= \frac{(x-y)^2}{(x^2-y^2)^2}-\frac{y^2-x^2}{(x^2-y^2)^2}-\frac{(x+y)^2}{(x^2-y^2)^2}\\ &= \frac{(x-y)^2-y^2+x^2-(x+y)^2}{(x^2-y^2)^2}\\ &= \frac{x^2-2xy+y^2-y^2+x^2-x^2-2xy-y^2}{(x^2-y^2)^2}\\ &= \frac{x^2-4xy-y^2}{(x^2-y^2)^2}.\end{aligned}$$

Exercise XL.

Simplify—

(1.) $\frac{a}{b}+\frac{b}{a}$. (2.) $\frac{x}{2a}+\frac{1}{a^2}$. (3.) $\frac{x}{3}+\frac{1}{2a}+\frac{2}{4a^2}$.

(4.) $\frac{1}{x}-\frac{2}{3x^2}$. (5.) $\frac{2}{x}+\frac{3a}{x^2}-\frac{1}{4x^3}$. (6.) $\frac{x-1}{2x}-\frac{x-3}{x^2}$.

(7.) $\frac{2a-3b}{3}-\frac{3a-2b}{4}-\frac{a-3b}{5}$. (8.) $\frac{a+b}{a-b}+\frac{a-b}{a+b}$.

(9.) $\frac{a+b}{a-b}-\frac{a-b}{a+b}$. (10.) $\frac{2a}{1-a^2}+\frac{1}{1+a}$.

(11.) $\frac{b}{a}+\frac{b}{a+b}+\frac{a^2}{a^2+ab}$. (12.) $\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}$.

(13.) $\frac{x+1}{x^2+x+1}+\frac{x-1}{x^2-x+1}$. (14.) $\frac{1}{2(x-1)}-\frac{1}{2(x+1)}-\frac{1}{x^2}$.

(15.) $\frac{1}{2x-4y}-\frac{1}{2x+4y}+\frac{x}{x^2-4y^2}$.

(16.) $\frac{2}{x-1}-\frac{1}{x+1}-\frac{x+3}{x^2+1}$.

(17.) $\frac{x+y}{x^2+xy+y^2}+\frac{x-y}{x^2-xy+y^2}+\frac{2y^3}{x^4+x^2y^2+y^4}$.

(18.) $\frac{x-1}{x^2-x+1}+\frac{1}{x+1}+\frac{x+1}{x^2+x+1}+\frac{1}{x-1}$.

(19.) $\frac{2}{(x-1)(x+2)}-\frac{2}{(x-1)(x-2)}+\frac{1}{(x-2)(x+2)}$.

(20.) $\frac{ab}{(a-b)(b-c)},+\frac{ac}{(a-c)(c-b)}$.

(21.) $\frac{a}{(x-b)(x-c)}+\frac{b}{(x-c)(x-a)}+\frac{c}{(x-a)(x-b)}$

$$+\frac{a^2+b^2+c^2}{(x-a)(x-b)(x-c)}.$$

(22.) $\frac{x-a}{(b-a)(c-a)}+\frac{x-b}{(a-b)(c-b)}+\frac{x-c}{(a-c)(b-c)}$.

(23.) $\frac{a^2-bc}{(a-b)(a-c)}+\frac{b^2-ac}{(b-a)(b-c)}+\frac{c^2-ab}{(c-b)(c-a)}$.

135. A *mixed quantity* is the sum of a whole expression and a fraction; as, for example,

$$a+\frac{b}{c}, \quad x^2-1+\frac{3x}{x^3+5}, \quad x+3-\frac{x-1}{x^2+2}.$$

136. *A mixed quantity may be expressed as a fraction by considering a whole expression as a fraction whose denominator is unity;* and conversely, *a fraction may be expressed as a mixed quantity when part of its numerator is a multiple of its denominator.*

Examples.

(1.) $2x+\frac{1}{a}=\frac{2x}{1}+\frac{1}{a}=\frac{2ax+1}{a}$.

(2.) $x-1+\frac{1}{x+1}=\frac{x-1}{1}+\frac{1}{x+1}=\frac{x^2}{x+1}$.

(3.) $\frac{x^4+1}{x^2+1}-x^2+1=\frac{x^4+1}{x^2+1}-\frac{x^2-1}{1}=\frac{2}{x^2+1}$.

It will be observed here that $-x^2+1=-(x^2-1)=-\frac{x^2-1}{1}$.

(4.) Express $\frac{2x^2-3x+4}{x+1}$ as a mixed quantity.

On dividing $2x^2-3x+4$ by $x+1$ we get quotient $2x-5$ and remainder 9.

$$\therefore \frac{2x^2-3x+4}{x+1}=2x-5+\frac{9}{x+1}.$$

(5.) Express $\frac{x^3+x^2-2x+3}{x^2-x+1}$ as a mixed quantity.

In this case the quotient is $x+2$, and remainder $-x+1$.

$$\therefore \frac{x^3+x^2-2x+3}{x^2-x+1}=x+2+\frac{-x+1}{x^2-x+1}.$$

$$=x+2-\frac{x-1}{x^2-x+1}.$$

Here the + before the fraction is changed to − by changing at the same time the signs of all the terms in the numerator $-x+1$.

Exercise XLI.

Simplify—

(1.) $\frac{3}{x}+5$. (2.) $\frac{x}{a^2}-1$. (3.) $2-\frac{x}{a}$.

(4.) $\frac{a}{x}+a-1$. (5.) $\frac{3a}{x}-a+1$. (6.) $2+\frac{x-1}{x^2}$.

(7.) $3-\frac{x^2-1}{x^2}$. (8.) $4-\frac{x^2-2x+3}{x^3}$.

(9.) $\frac{4}{x-1}+3$. (10.) $\frac{a}{a-b}-a$. (11.) $2-\frac{a-b}{a+b}$.

(12.) $4x-1-\frac{2x+1}{x+3}$. (13.) $x^2+xy+y^2-\frac{xy-y^3}{x-y}$.

Express as mixed quantities—

(14.) $\frac{ab+c}{a}$. (15.) $\frac{ax-1}{x}$. (16.) $\frac{2-3x}{3x}$.

(17.) $\frac{2x^2-x+3}{x}$. (18.) $\frac{4-3x+6x^2}{3x}$.

(19.) $\frac{x^2+3x+4}{x^2-1}$. (20.) $\frac{5x-6}{x-1}$. (21.) $\frac{4+2x}{1+x}$.

(22.) $\frac{x^3-3x^2+2}{x^2-3}$. (23.) $\frac{6x^2-4x+5}{2x^2-x+1}$. (24.) $\frac{x^5-x+5}{x^3+1}$.

Multiplication.

137. To denote that two or more fractions, or a fraction and a mononomial whole quantity, are to be multiplied together, they are written in a row with the multiplication sign $\times$, or $\cdot$ (dot), between them.

Thus $\frac{a}{b}\times\frac{c}{d}$, or $\frac{a}{b}\cdot\frac{c}{d}$ denotes that $\frac{a}{b}$ is to be multiplied by $\frac{c}{d}$; $\frac{x-1}{a^2}\cdot\frac{2x^2-3}{x^3-1}$ denotes the product of $\frac{x-1}{a^2}$ and $\frac{2x^2-3}{x^3-1}$; $\frac{2a-x}{b}\times-3ax$ denotes the product of $\frac{2a-x}{b}$ and $-3ax$.

Sums and differences when they are factors must be enclosed in brackets.

Thus, $\frac{a}{a^3-x^3}(a^2-ax+x^2)$ denotes the product of $\frac{a}{a^3-x^3}$ and a^2-ax+x^2; $\frac{x}{x+a}\left(x+\frac{a}{x}\right)$ denotes the product of $\frac{x}{x+a}$ and $x+\frac{a}{x}$; $\frac{x-2}{x^2-1}\left(\frac{x}{2}+\frac{3}{x}-\frac{1}{x^2}\right)$ denotes the product of $\frac{x-2}{x^2-1}$ and $\frac{x}{2}+\frac{3}{x}-\frac{1}{x^2}$.

138. Multiplication of fractions is performed according to the following rule:—

The product of any number of fractions is a fraction whose numerator is the product of their numerators and denominator the product of their denominators.

A whole quantity is to be considered as a fraction whose denominator is unity; and sums and differences of fractions and mixed quantities must first be reduced to fractions.

Examples.

$$(1.)\ \frac{x^2-a^2}{2ax}\cdot\frac{ax}{x+a}=\frac{(x^2-a^2)ax}{2ax(x+a)}=\frac{x-a}{2}.$$

Factors common to any numerator and any denominator may be struck out. Thus ax which is common to the first denominator and second numerator, and $x+a$ which is common to the first numerator and second denominator, may be struck out before multiplying, and the result will be in lowest terms. When possible, therefore, the component factors of the several numerators and denominators should be obtained.

$$(2.)\ \frac{4a^3-4ax^2}{3bc^2-3bx^2}\cdot\frac{bc+bx}{a^2-ax}=\frac{4a(a+x)(a-x)}{3b(c+x)(c-x)}\cdot\frac{b(c+x)}{a(a-x)}$$
$$=\frac{4(a+x)}{3(c-x)}.$$

Here a, $a-x$, b, and $c+x$ are struck out, being common to the numerators and denominators.

(3.) $$\frac{x}{(x+6)(x+7)}(x^2-49)=\frac{x}{(x+6)(x+7)}\cdot\frac{(x+7)(x-7)}{1}$$
$$=\frac{x(x-7)}{x+6}.$$

(4.) $$\left(3-\frac{2x^2-4}{x^2-1}\right)\frac{x-1}{x^2+1}=\frac{3x^2-3-2x^2+4}{x^2-1}\cdot\frac{x-1}{x^2+1}$$
$$=\frac{x^2+1}{(x+1)(x-1)}\cdot\frac{x-1}{x^2+1}$$
$$=\frac{1}{x+1}.$$

EXERCISE XLII.

Simplify—

(1.) $\frac{2x}{3y}\times\frac{6ay}{5x^2}$.

(2.) $\frac{a^2}{bc}\cdot\frac{b^2}{ca}\cdot\frac{c^2}{ab}$.

(3.) $\frac{b^2c}{yz^2}\cdot\frac{c^2a}{zx^2}\cdot\frac{a^2b}{xy^2}$.

(4.) $\frac{x}{1-x}\cdot\frac{y}{1+x}$.

(5.) $\frac{a}{a-b}\cdot\frac{a^2-b^2}{ab}$.

(6.) $\frac{a^3-b^3}{a^3+b^3}\cdot\frac{a+b}{a-b}$.

(7.) $\frac{a^3-b^3}{a^3+b^3}\cdot\frac{a^2-ab+b^2}{a^2-b^2}$.

(8.) $\frac{a^3b-ab^3}{(a-b)^3}\cdot\frac{a^3-b^3}{a^3b^3}$.

(9.) $\frac{1}{x+1}\cdot\frac{x+1}{x^2+1}\cdot\frac{x^2+1}{x^3+1}$.

(10.) $\frac{x^2-1}{x^3+1}\cdot\frac{x^3-1}{x^2+1}\cdot\frac{x^4-1}{x^6-1}$.

(11.) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)xyz$.

(12.) $\frac{ax}{a+x}\left(\frac{x}{a}-\frac{a}{x}\right)$.

(13.) $\left(b+\frac{a^2}{b}\right)\left(a-\frac{b^2}{a}\right)$.

(14.) $\frac{x^2+xy}{x^2+y^2}\left(\frac{x}{x-y}-\frac{y}{x+y}\right)$.

(15.) $\left(\frac{a}{a^2-ab+b^2}-\frac{1}{a}\right)\frac{1}{a-b}$.

(16.) $\frac{a}{bx}\left(b+\frac{bx}{a}\right)\left(1-\frac{a}{a+x}\right)$.

(17.) $\left(\frac{1}{x^2}-\frac{1}{xy}+\frac{1}{y^2}\right)\frac{x^3y^3}{x^3+y^3}$.

(18.) $\left(\frac{a+b}{a-b}+\frac{a^2+b^2}{a^2-b^2}\right)\frac{a+b}{a^3-b^3}$.

Division.

139. To denote that one fraction is to be divided by another, they are written in a row with the sign $\div$ between them. The same notation is employed when the dividend or divisor is a mononomial whole quantity.

Sums and differences when they are the object of division must be enclosed in brackets.

Thus, $\frac{2}{x}\div\frac{x}{a}$ denotes that $\frac{2}{x}$ is to be divided by $\frac{x}{a}$;

$\frac{4a}{x^2}\div 3c$ „ „ $\frac{4a}{x^2}$ „ „ $3c$;

$\frac{a}{b}\div(2a-1)$ „ $\frac{a}{b}$ „ „ $2a-1$;

$\frac{x}{y}\div\left(x^2-\frac{3x}{a}\right)$ „ „ $\frac{x}{y}$ „ „ $x^2-\frac{3x}{a}$;

$(x^2-5)\div\frac{2a}{x}$ „ „ x^2-5 „ „ $\frac{2a}{x}$.

The same thing may also be denoted by writing the quantities which are the objects of division in the form of a fraction.

Thus, $\dfrac{\frac{2x}{3y}}{\frac{x}{a}}=\frac{2x}{3y}\div\frac{x}{a}$; $\dfrac{\frac{a}{b}}{2a-1}=\frac{a}{b}\div(2a-1)$;

$$\dfrac{\frac{x}{y}}{x^2-\frac{3x}{a}}=\frac{x}{y}\div\left(x^2-\frac{3x}{a}\right);\quad \dfrac{x^2-5}{\frac{2a}{x}}=(x^2-5)\div\frac{2a}{x}.$$

140. When the product of two quantities is unity, each is said to be the *reciprocal* of the other.

Thus, since $a\times\frac{1}{a}=1$, $\frac{a}{b}\cdot\frac{b}{a}=1$, $\frac{x-1}{2x-3}\cdot\frac{2x-3}{x-1}=1$, it follows that

$\frac{1}{a}$ is the reciprocal of a, and a of $\frac{1}{a}$;

$\frac{a}{b}$ " " $\frac{b}{a}$, " $\frac{b}{a}$ " $\frac{a}{b}$;

$\frac{x-1}{2x-3}$ " " $\frac{2x-3}{x-1}$, " $\frac{2x-3}{x-1}$ " $\frac{x-1}{2x-3}$.

141. Division of fractions is performed according to the following rule:—

The quotient of one fraction divided by another is the product of the former and the reciprocal of the latter.

A whole quantity is to be considered as a fraction whose denominator is unity; and sums and differences of fractions and mixed quantities must first be reduced to fractions.

Examples.

(1.) $\frac{4a^2b}{5xy^2} \div \frac{2ab^2}{15x^2y} = \frac{4a^2b}{5xy^2} \cdot \frac{15x^2y}{2ab^2} = \frac{6ax}{by}$.

(2.) $$\frac{a^2}{a^3-x^3} \div \frac{a^2+ax}{a^2+ax+x^2} = \frac{a^2}{a^3-x^3} \cdot \frac{a^2+ax+x^2}{a^2+ax}$$

$$= \frac{a^2}{(a-x)(a^2+ax+x^2)} \cdot \frac{a^2+ax+x^2}{a(a+x)}$$

$$= \frac{a}{a^2-x^2}.$$

(3.) $$\left(\frac{x}{a}+\frac{y}{b}\right) \div \left(\frac{x}{a}-\frac{y}{b}\right) = \frac{bx+ay}{ab} \div \frac{bx-ay}{ab}$$

$$= \frac{bx+ay}{ab} \cdot \frac{ab}{bx-ay}$$

$$= \frac{bx+ay}{bx-ay}.$$

(4.) $$\left(\frac{1}{x^2}-\frac{1}{xy}+\frac{1}{y^2}\right) \div (x^3+y^3) = \left(\frac{1}{x^2}-\frac{1}{xy}+\frac{1}{y^2}\right)\frac{1}{x^3+y^3}$$

$$= \frac{y^2-xy+x^2}{x^2y^2} \cdot \frac{1}{(x+y)(x^2-xy+y^2)}$$

$$= \frac{1}{x^2y^2(x+y)}.$$

Exercise XLIII.

Simplify—

(1.) $\frac{2ab}{5xy} \div \frac{3x}{10a}$.

(2.) $\frac{4a^2x^3}{5b^2y} \div \frac{3ax}{5by^2}$.

(3.) $\frac{1}{x+1} \div \frac{1}{x-1}$.

(4.) $\frac{a}{a+x} \div \frac{a}{a-x}$.

(5.) $\frac{a^3-b^3}{a^3+b^3} \div \frac{a-b}{a+b}$.

(6.) $\frac{a^4-b^4}{a^2+2ab+b^2} \div \frac{a^2-ab}{a+b}$.

(7.) $\left(\frac{1}{x}+\frac{1}{a}\right) \div \frac{1}{x}$.

(8.) $\left(\frac{a+b}{a-b}+\frac{a^2+b^2}{a^2-b^2}\right) \div \frac{a^3-b^3}{a+b}$.

(9.) $\dfrac{\frac{a+p}{a-p}+\frac{a-p}{a+p}}{\frac{a+p}{a-p}-\frac{a-p}{a+p}}$.

(10.) $\dfrac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{2(a^2+x^2)}$.

(11.) $\dfrac{\frac{x}{x-y}+\frac{y}{x+y}}{\frac{x}{x-y}-\frac{y}{x+y}}$.

(12.) $\dfrac{\frac{1}{a^2}-\frac{1}{ax}+\frac{1}{x^2}}{\frac{1}{a^3}+\frac{1}{x^3}}$.

(13.) $\dfrac{\frac{x}{4}-\frac{x}{5}}{\frac{x}{4}+\frac{x}{5}}+\dfrac{\frac{x}{5}-\frac{x}{6}}{\frac{x}{5}+\frac{x}{6}}$.

(14.) $\dfrac{\frac{x}{8}-\frac{x}{10}}{\frac{x}{8}+\frac{x}{10}}-\dfrac{\frac{x}{10}-\frac{x}{12}}{\frac{x}{10}+\frac{x}{12}}$.

CHAPTER XV.

SIMPLE EQUATIONS (continued).

142. We shall give in this chapter some examples of equations involving fractions with literal denominators. Such equations may be cleared of fractions by the rule already given in Art. 74. In some cases before applying this rule it will be found more advantageous to simplify parts of the equation separately.

Examples.

(1.) Solve $$\frac{x-a}{b}=\frac{x-b}{a}.$$

Multiplying by ab, the L.C.M. of the denominators, we get

$$(x-a)a=(x-b)b.$$

Clearing of brackets and transposing,

$$ax-bx=a^2-b^2.$$

Collecting coefficients of x,

$$(a-b)x=a^2-b^2.$$

Dividing by $a-b$,

$$x=\frac{a^2-b^2}{a-b}=a+b.$$

(2.) Solve $$\frac{3x-1}{2x-1}-\frac{4x-2}{3x-2}=\tfrac{1}{6}.$$

Multiplying by the L.C.M. $6\,(2x-1)\,(3x-2)$,

$$6(3x-1)(3x-2)\,.\,6(4x-2)(2x-1)=(2x-1)(3x-2).$$

Clearing of brackets,

$$54x^2-54x+12-48x^2+48x-12=6x^2-7x+2.$$

Transposing,

$$54x^2-48x^2-6x^2-54x+48x+7x=2-12+12,$$
$$\therefore\ x=2.$$

(3.) Solve $$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-4}{x-5}-\frac{x-5}{x-6}.$$

Simplifying the sides separately,

$$\frac{(x-1)(x-3)-(x-2)^2}{(x-2)(x-3)}=\frac{(x-4)(x-6)-(x-5)^2}{(x-5)(x-6)}.$$

Clearing the numerators of brackets,

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-5)(x-6)}.$$

Multiplying by the L.C.M. $(x-2)(x-3)(x-5)(x-6)$, clearing of brackets, and solving,

$$x=4.$$

Exercise XLIV.

Solve—

(1.) $\frac{12}{x}+\frac{1}{12x}=\frac{29}{24}.$

(2.) $\frac{42}{x-2}=\frac{35}{x-3}.$

(3.) $\frac{16}{3x-4}=\frac{27}{5x-6}.$

(4.) $\frac{45}{2x+3}=\frac{57}{4x-5}.$

(5.) $\frac{x-1}{x-2}=\frac{7x-21}{7x-26}.$

(6.) $\frac{2x-6}{3x-8}=\frac{2x-5}{3x-7}.$

(7.) $\frac{x}{x+1}+2=\frac{3x}{x+2}.$

(8.) $\frac{2x-3}{2x+1}+\frac{2x-1}{2x+3}=2.$

(9.) $\frac{5x-3}{x+1}-\frac{2x+3}{2x+1}=4.$

(10.) $\frac{6x+13}{15}-\frac{2x}{5}=\frac{3x+5}{5x-25}.$

(11.) $\frac{x-5}{x}=\frac{1}{2}+\frac{x-5}{2x-5}.$

(12.) $\frac{x-14}{x}=\frac{2x-29}{2x-20}-\frac{1}{2x}.$

(13.) $\frac{2}{(2x+3)(x-5)}=\frac{3}{(3x-2)(x-11)}.$

(14.) $\dfrac{x^2+1}{4x^2-1}+\frac{1}{4}-\dfrac{x}{2x+1}=0.$

(15.) $\dfrac{x}{x-1}-\dfrac{x-1}{x-2}=\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}.$

(16.) $\dfrac{x}{6-x}-\dfrac{x-1}{5-x}-\dfrac{x-2}{4-x}=1.$

(17.) $\dfrac{x}{a}-\dfrac{x}{b}=c.$

(18.) $\dfrac{x-a}{b}+\dfrac{x-b}{a}=\dfrac{a}{b}+\dfrac{b}{a}.$

(19.) $\dfrac{x}{a}+\dfrac{a}{b}=\dfrac{x}{b}+\dfrac{b}{a}.$

(20.) $\dfrac{x-a}{a+b}+\dfrac{x-b}{a-b}=1.$

(21.) $\dfrac{b}{ax}-\dfrac{a}{bx}=b^2-a^2.$

(22.) $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}=\dfrac{x+a}{x-b}+\dfrac{x+b}{x-a}.$

CHAPTER XVI.

PROBLEMS (*continued*).

143. We shall give in this chapter some examples which are more difficult than those in Chapter IX.

Examples.

(1.) If A can perform a given work in 60 days, and B in 40 days, in how many days will A and B, working together, be able to perform it?

Let w denote the work to be done, and x the required number of days. Then

amount of work done by A in one day $=\frac{w}{60}$;

" " B " $=\frac{w}{40}$;

" " A and B " $=\frac{w}{x}$.

$$\therefore \frac{w}{x}=\frac{w}{60}+\frac{w}{40}.$$

Divide by w, $\frac{1}{x}=\frac{1}{60}+\frac{1}{40}$.

Multiply by $120x$, $120=2x+3x$.

$$\therefore x=24.$$

(2.) At what time between 5 and 6 is the minute hand of a watch 5 minute divisions behind the hour hand?

Let $x=$ the number of minute divisions between the hour hand and 5; then $5-x=$the number of minute divisions between the minute hand and 5. But the number of minute divisions between 12 and 5 is 25; therefore the number of minute divisions between 12 and the minute hand is $25-(5-x)=20+x$.

The hour hand thus moves over x minute divisions while the minute hand moves over $20+x$; and since the latter moves 12 times faster than the former, it follows that

$$20+x=12x.$$

$$\therefore x=1\tfrac{9}{11}.$$

Hence the required time is $12x=21\tfrac{9}{11}$ minutes past 5.

(3.) A grocer bought 200 lbs. of tea and 1000 lbs. of sugar, the price of the sugar being $\frac{1}{6}$ of that of the tea. He sold the tea at a profit of 40 per cent., and the sugar at a loss of $2\frac{1}{2}$ per cent., gaining on the whole \$45·50. What were his buying and selling prices?

Let the cost price of the sugar per lb. $=x$ dollars.

∴	„	„	tea „	$=6x$	„
Then	„	„	1000 lbs. of sugar	$=1000x$	dollars.
	„	„	200 „ tea	$=1200x$	„

The profit on the tea $=\dfrac{40}{100}\cdot 1200x=480x$

The loss „ sugar $=\dfrac{2\frac{1}{2}}{100}\cdot 1000x=25x$

$$\therefore 480x-25x=45{\cdot}50$$
$$x={\cdot}10$$

∴ the buying price of sugar is 10 cents, and the selling price $9\frac{3}{4}$ cents per lb.; the buying price of tea is 60 cents, and the selling price 84 cents per lb.

Exercise XLV.

(1.) I arrange 1024 men 8 deep in a hollow square: how many men will there be in each outer face?

(2.) A regiment containing 700 men is formed into a hollow square 5 ranks deep: how many men are there in the front rank?

(3.) A man has a number of cents which he tries to arrange in the form of a square; on the first attempt he has 130 over; when he increases the side of the square by 3 cents he has only 31 over. How many cents has he?

(4.) On a side of cricket consisting of 11 men, one-third more were bowled than run out, and 3 times as many run out as stumped; two were caught out. How many were bowled and run out, respectively?

(5.) Water expands 10 per cent. when it turns to ice. How much per cent. does ice contract when it turns to water?

(6.) A manufacturer adds to the cost price of goods 20 per cent. of it to give the selling price; afterwards, to effect a rapid sale, he deducts from the selling price of each article a discount of 10 per cent., and then obtains on each article a profit of 8 shillings. What was the cost price of each article?

(7.) A person invests £14,970 in the purchase of 3 per cents. at 90 and $3\frac{1}{4}$ per cents. at 97. His total income being £500, how much of each stock did he buy?

(8.) A and B join capital for a commercial enterprise, B contributing £250 more than A. If their profits amount to 10 per cent. on their joint capital, B's share of them is 12 per cent. on A's capital. How much does each contribute?

*(9.) In a concert room 800 persons are seated on benches of equal length. If there were 20 fewer benches, it would be necessary that two persons more should sit on each bench. Find the number of benches.

*(10.) A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he would have been 6 hours longer in performing the journey. Determine his rate of travelling.

(11.) An express train running from London to Wakefield (a distance of 180 miles) travels half as fast again as an

* These questions belong to Exercise XLVIII.

ordinary train, and performs the distance in two hours less time; find the rates of travelling.

(12.) A can do half as much work as B, B can do half as much as C, and together they can complete a piece of work in 24 days; in what time could each alone complete the work?

(13.) Three persons can together complete a piece of work in 60 days; and it is found that the first does $\frac{3}{4}$ of what the second does, and the second $\frac{4}{5}$ of what the third does: in what time could each alone complete the work?

(14.) What is the first time after 7 o'clock when the hour and minute hands of a watch are exactly opposite?

(15.) The hour is between 2 and 3 o'clock, and the minute hand is in advance of the hour hand by $14\frac{1}{2}$ minute spaces of the dial. What o'clock is it?

(16.) At what time between 3 and 4 o'clock is one hand of a watch exactly in the direction of the other hand produced?

(17.) The hands of a watch are at right angles to each other at 3 o'clock: when are they next at right angles?

(18.) How much water must be mixed with 60 gallons of spirit which cost £1 per gallon, that on selling the mixture at 22s. per gallon a gain of £17 may be made?

(19.) How much water must be mixed with 80 gallons of spirit bought at 15s. per gallon, so that on selling the mixture at 12s. per gallon there may be a profit of 10 per cent. on the outlay?

(20.) If 16 oz. of sea-water contain 0·8 oz. of salt, how much pure water must be added that 16 oz. of the mixture may contain only 0·1 oz. of salt?

(21.) I have a bar of metal containing 80 per cent. pure gold, which weighs 30 grains: how much must I add to this of metal containing 90 per cent. pure gold, in order that the mixture may contain 87 per cent.?

(22.) How much silver must I add to 2 lbs. 6 oz. of an

alloy of silver and gold containing 91·7 per cent. of pure gold, in order that the mixture may contain 84 per cent. of gold?

(23.) A person started at a certain pace to walk to a railway station 3 miles off, intending to arrive at a certain time; but, after walking a mile, he was detained 10 minutes, and was in consequence obliged to walk the rest of the way a mile an hour faster. At what pace did he start?

(24.) A person started at the rate of 3 miles an hour to walk to a railway station in order to catch a train, but after he had walked $\frac{1}{3}$ of the distance he was detained 15 minutes, and was obliged in consequence to walk the rest of the way at the rate of 4 miles an hour. How far off was the station?

(25.) A wins the 200 yard race in $28\frac{1}{2}$ seconds, B the consolation stakes (same distance) in 30 seconds: how many yards ought A to give B in a handicap?

(26.) A wins a mile race with B in 5′ 19″. B runs at a uniform pace all the way; A runs at $\frac{10}{11}$ of B's pace for the greater part of the distance, and then doubles his pace, winning by a second: how far did A run before changing his pace?

(27.) A boy swam half a mile down a stream in 10 minutes; without the aid of the stream it would have taken him a quarter of an hour. What was the rate of the stream per hour; and how long would it take him to return against it?

(28.) A contractor undertook to build a house in 21 days, and engaged 15 men to do the work. But after 10 days he found it necessary to engage 10 men more, and then he accomplished the work one day too soon. How many days behindhand would he have been if he had not engaged the 10 additional men?

(29.) Two crews row a match over a four-mile course; one pulls 42 strokes a minute, the other 38, and the latter does the distance in 25 minutes; supposing both crews to row uniformly, and 40 strokes of the former to be equivalent to 36 of the latter, find the position of the losing boat at the end of the race.

CHAPTER XVII.

QUADRATIC EQUATIONS.

144. WE have already defined *Quadratic Equations* in Art. 69. They are further called *adfected* or *pure*, according as the term involving the first power of the unknown quantity does, or does not, appear.

Thus, $4x^2-5x+7=0$, $x^2+6x-3=0$, $x^2-3x=0$, are adfected quadratics; $3x^2-8=0$, $x^2+6=0$, are pure quadratics.

145. I. PURE QUADRATICS are solved *by transposition of terms and extraction of the square root.*

Examples.

(1.) Solve $x^2-4=0$.

Transposing, $\qquad x^2=4.$

Since the square root of a positive quantity is either + or −, we have, extracting the square root,

$$x=\pm 2.$$

Thus the two roots are $+2$, -2.

(2.) Solve $x^2+5=\frac{10}{3}x^2-16$.

Clearing of fractions,

$$3x^2+15=10x^2-48.$$

Transposing and dividing by -7,

$$x^2=9.$$

$$\therefore x=\pm 3.$$

Thus the roots are $+3$ and -3.

146. II. ADFECTED QUADRATICS may be solved by one of the following three rules:—

(i.) *When the equation is in the form of the product of two factors, each containing the unknown, equated to zero, the solution is effected by equating to zero each factor in turn.*

Examples.

(1.) Solve $(x+1)(2x-3)=0$.

Since, when the product of two factors vanishes, one or other must be zero, we have

$$\text{either } x+1=0, \text{ and } \therefore x=-1;$$

$$\text{or } 2x-3=0, \text{ and } \therefore x=\tfrac{3}{2}.$$

Thus the two roots are -1 and $\frac{3}{2}$.

(2.) Solve $x^2-5x=0$.

Factoring, $x(x-5)=0$.

$$\therefore \text{ either } x=0,$$

$$\text{or } x-5=0, \text{ and } \therefore x=5.$$

Thus the roots are 0 and 5.

Whenever, as in this case, the terms of an equation are divisible by the unknown x, we can infer that one root is zero.

(3.) Solve $(2x-5)(ax-4b)=0$.

Here either $2x-5=0$, and $\therefore x=\frac{5}{2}$;

$$\text{or } ax-4b=0, \text{ and } \therefore x=\frac{4b}{a}.$$

Thus the roots are $\frac{5}{2}$ and $\frac{4b}{a}$.

EXERCISE XLVI.

(1.) $x^2-36=0$. (2.) $5x^2=45$. (3.) $\frac{x^2}{3}=27$.

(4.) $2(x^2-7)+3(x^2-11)=33$.

(5.) $\frac{1}{2}(x^2+4)+\frac{1}{7}(x^2+3)=x^2+1$. (6.) $\frac{6}{x^2-2}=\frac{7}{x^2+2}$.

(7.) $x^2=3x$. (8.) $\frac{x^2}{2}+6x=0$. (9.) $\frac{1}{3}(x^2-4x)=5x$.

(10.) $x^2-\frac{7x}{2}=0$. (11.) $4x^2+\frac{x}{3}=0$.

(12.) $x^2-\frac{2x}{3}=2x^2+x$. (13.) $(x-3)(x-5)=0$.

(14.) $(x+5)(x-7)=0$. (15.) $(x+1)(x+3)=0$.

(16.) $(2x-1)(3x-4)=0$. (17.) $(3x-5)(2x+7)=0$.

(18.) $(5x+6)(6x+7)=0$. (19.) $(ax-b)(cx+d)=0$.

(20.) $x^2-ax=ax-x^2$.

147. When the quadratic is not in a form adapted for applying rule (i.), it may be solved by either of the following rules:—

(ii.) *Having transposed the unknowns separately to one side, make the coefficient of* x^2 *unity by division (if necessary). Then add the square of* $\frac{1}{2}$ *the coefficient of* x, *and the solution is effected by the extraction of the square root of both sides.*

148. (iii.) *Having cleared the equation of fractions (if necessary), and transposed the unknowns separately to one side, multiply both sides by 4 times the coefficient of* x^2, *and add the square of the coefficient of* x. *The solution is then effected by the extraction of the square root of both sides.*

Examples.

(1.) Solve $x^2-12x+35=0$.

By rule (ii.), transposing,

$$x^2-12x=-35.$$

Adding the square of one-half 12,

$$x^2-12x+6^2=36-35=1.$$

Extracting the square root,

$$x-6=\pm 1;$$

that is, $x-6=1$, and $\therefore\ x=7$,

or $x-6=-1$, and $\therefore\ x=5$.

Thus the roots are 7 and 5.

(2.) Solve $\qquad 2x^2+5x-3=0.$

By rule (ii.), transposing and dividing by 2,

$$x^2+\tfrac{5}{2}x=\tfrac{3}{2}.$$

Adding the square of one-half $\frac{5}{2}$,

$$x^2+\tfrac{5}{2}x+\left(\tfrac{5}{4}\right)^2=\tfrac{25}{16}+\tfrac{3}{2}=\tfrac{49}{16}.$$

Extracting the square root,

$$x+\tfrac{5}{4}=\pm\tfrac{7}{4}.$$

$$\therefore\ x=\frac{-5\pm 7}{4}=\tfrac{1}{2},\ \text{or } -3.$$

Thus the roots are $\frac{1}{2}$ and -3.

(3.) Solve $\qquad x^2+px+q=0.$

By (ii.), transposing,

$$x^2+px=-q.$$

Adding the square of one-half p,

$$x^2+px+\left(\frac{p}{2}\right)^2=\frac{p^2}{4}-q=\frac{p^2-4q}{4}.$$

Extracting the square root,

$$x+\frac{p}{2}=\pm\frac{\sqrt{p^2-4q}}{2}.$$

$$\therefore\ x=\frac{-p\pm\sqrt{p^2-4q}}{2}.$$

Thus the roots are

$$\frac{-p+\sqrt{p^2-4q}}{2} \text{ and } \frac{-p-\sqrt{p^2-4q}}{2}.$$

(4.) Solve $2x^2+5x=3$ by rule (iii.).

Multiplying by $4\times 2=8$,

$$16x^2+40x=24.$$

Adding the square of 5,

$$16x^2+40x+5^2=25+24=49.$$

Extracting the square root,

$$4x+5=\pm 7.$$

$$\therefore\ x=\tfrac{1}{2}, \text{ or } -3.$$

(5.) Solve $ax^2+bx+c=0$.

By (iii.), transposing,

$$ax^2+bx=-c.$$

Multiplying by $4a$ and adding b^2,

$$4a^2x^2+4abx+b^2=b^2-4ac.$$

Extracting the square root,

$$2ax+b=\pm\sqrt{b^2-4ac}.$$

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Thus the roots are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Exercise XLVII.

(1.) $x^2-6x+8=0$.

(2.) $x^2-4x-5=0$.

(3.) $x^2+4x-21=0$.

(4.) $2x^2-5x+2=0$.

(5.) $1-x^2=\frac{3x}{2}$.

(6.) $\frac{3x}{4}=x^2+\frac{1}{8}$.

(7.) $4x^2-4x=15.$

(8.) $6x^2-11x+4=0.$

(9.) $\frac{x^2}{2}-x=12.$

(10.) $x+\frac{1}{x}=\frac{13}{6}.$

(11.) $x-\frac{x^3-8}{x^2+5}=2.$

(12.) $1+\frac{7}{5x}=\frac{86}{x^2}.$

(13.) $\frac{21x^3-16}{3x^2-4}=7x+5.$

(14.) $\frac{x}{x+60}=\frac{7}{3x-5}.$

(15.) $\frac{2-x}{1-x}+\frac{1-x}{2-x}=\frac{13}{6}.$

(16.) $\frac{2x-3}{2x+1}-\frac{x-3}{3x+2}=1$

(17.) $x=5-\frac{1}{x-3}.$

(18.) $\frac{3x-2}{2x-5}-\frac{2x-5}{3x-2}=\frac{8}{3}.$

(19.) $\frac{x+4}{x}-\frac{5(x+2)}{x-6}=8.$

(20.) $\frac{x+2}{x-1}-\frac{7}{3}=\frac{4-x}{2x}.$

(21.) $(x-3)^2-3(x-2)(x-7)=21.$

(22.) $x^2-(a+b)x+ab=0.$

CHAPTER XVIII.

PROBLEMS.

149. When the Algebraical statement of a problem leads to a quadratic equation, the unknown quantity will be one of the roots. In some cases either root may be taken, but it will generally be found that one of the roots must be rejected as being inconsistent with the conditions of the particular question proposed.

Examples.

(1.) A person laid out a certain sum of money in goods which he sold again for \$24, and lost as much per cent. as he laid out. Find out how much he laid out.

Let x = number of dollars laid out.

$\therefore x-24=$ " " lost.

But the loss is also x per cent. of $x=\frac{x}{100}\times x=\frac{x^2}{100}$.

$$\therefore \frac{x^2}{100}=x-24.$$

$$x^2-100x=-2400.$$

$$x=40 \text{ or } 60.$$

The amount laid out was, therefore, \$40 or \$60. Thus both roots satisfy the conditions of the problem.

(2.) A person buys a certain number of shares for as many dollars per share as he buys shares; after they have risen as many cents per share as he has shares, he sells and gains \$100. How many shares did he buy?

Let $x=$ the number of shares bought, the price of which at x dollars per share is x^2 dollars.

The rise being x cents or $\frac{x}{100}$ dollars, the price for which he afterwards sells the x shares at $x+\frac{x}{100}$ dollars per share is $\left(x+\frac{x}{100}\right)x$. But the gain is \$100.

$$\therefore\ \left(x+\frac{x}{100}\right)x-x^2=100;$$

$$x^2=10000.$$

$$\therefore\ x=+100, \text{ or } -100.$$

As the negative root would not answer the conditions of the problem, it must be rejected. The answer is, therefore, 100.

Exercise XLVIII.

(1.) A rectangular room which contains 1800 square feet is twice as long as it is broad: find its dimensions.

(2.) Divide 20 into two parts whose product shall be 91.

(3.) Find a number whose square increased by 20 is 12 times as great as the number itself.

(4.) Divide 15 into two parts such that their product shall be 4 times their difference.

(5.) By what number must I divide 24 in order that the sum of the divisor and quotient may be 10?

(6.) Find three consecutive numbers such that the square of the greater shall be equal to the sum of the squares of the other two.

(7.) A ladder 34 feet long just reached a window of a house, when placed in such a position that the height of the window above the ground exceeded the distance of the foot of the ladder from the wall by 14 feet. Find the height of the window.

(8.) A horse is sold for £24, and the number expressing the profit per cent. also expresses the cost price of the horse: what did he cost?

(9.) An article is sold for £9 at a loss of as much per cent. as it is worth. Find its value.

(10.) A and B start together for a walk of 10 miles; A walks $1\frac{1}{2}$ miles an hour faster than B, and arrives $1\frac{1}{2}$ hours sooner than he does: at what rate did each walk?

(11.) After selling a part of an estate, and the same part of the remainder, I find I have left nine-tenths of the part first sold: what part did I sell at first?

(12.) An uncle leaves 14,000 dollars among his nephews and nieces, but 3 of them having died in his lifetime, the others received 600 dollars apiece more: how many nephews and nieces were there originally?

(13.) A number is composed of two digits, the first of which exceeds the second by unity, and the number itself falls short of the sum of the squares of its digits by 26. What is the number?

(14.) The sides of a rectangle are 12 and 20 feet: what is the breadth of the border which must be added all round that the whole area may be 384 square feet?

(15.) One hundred and ten bushels of coals are distributed among a certain number of poor persons; if each had received one bushel more, then he would have received as many bushels as there were persons. How many persons were there?

(16.) A sum of £23 is divided among a certain number of persons; if each one had received 3 shillings more, he would have received as many shillings as there were persons. How many persons were there?

(17.) A company at an inn had £7 4*s.* to pay, but before the bill was settled 3 of them left the room, and then those who remained had 4*s.* apiece more to pay than before; of how many did the company consist?

(18.) A person rents a certain number of acres of pasture land for £70; he keeps 8 acres in his own possession, and sublets the remainder at 5*s*. an acre more than he gave, and thus covers his rent and has £2 over. How many acres were there?

(19.) An officer can form the men in his battalion into a solid square, and also into a hollow square 12 deep; if the front in the latter formation exceed the front in the former by 3, find the number of men in the battalion.

CHAPTER XIX.

SIMULTANEOUS EQUATIONS.

150. If two unknowns are to be determined, there must be two independent equations. These equations are called *simultaneous* equations, because the same values of the unknowns x and y must be substituted in both equations. Thus if

$$2x-y=9,$$
$$2y-x=3,$$

the only values which satisfy both these equations at the same time are $x=7$, $y=5$.

151. It must be borne in mind that there is an infinite number of values which will satisfy either equation separately.

Thus, in the equation $2x-y=9$,

$$\text{if } x=1,\ 2-y=9, \text{ and } \therefore y=-7;$$
$$\text{if } x=2,\ 4-y=9, \text{ and } \therefore y=-5;$$
$$\text{if } x=10,\ 20-y=9, \text{ and } \therefore y=11;$$

and so on.

152. If three unknowns are to be determined, there must be three independent equations; and generally the number of unknowns must be the same as the number of independent equations connecting them.

153. The solution of simultaneous equations is effected by deducing from them other equations, each of which involves one unknown. This process is called *elimination,* and may be conducted according to one of the following methods:—

I. Substitution.

II. Comparison.

III. Cross Multiplication.

I. Method of Substitution.

154. *This method consists in finding from one equation the value of one unknown in terms of the other, and substituting the value so found in the second equation, which is thereby reduced to a simple equation in one unknown.*

For convenience of reference the given equations and others which arise in the process of solution are numbered (1), (2), (3), &c.

Example.

Solve $$x+y=3 \quad \ldots \quad (1),$$
$$2x+y=4 \quad \ldots \quad (2).$$

From (1) we find $y=3-x \quad \ldots \quad (3).$

Substituting this value of y in (2),

$$2x+3-x=4.$$
$$\therefore x=1.$$

Substituting this value of x in (3),

$$y=3-1=2.$$

Thus the solution is $x=1, \; y=2.$

II. Method of Comparison.

155. *This method consists in finding from each of the proposed equations the value of one and the same unknown in terms of the other, and equating the values so found.*

Example.

Solve $$7x-3y=19 \quad \ldots \quad (1),$$
$$4x+7y=37 \quad \ldots \quad (2).$$

From (1) we find

$$y=\frac{7x-19}{3} \quad \dots\dots (3);$$

and from (2)

$$y=\frac{37-4x}{7} \quad \dots\dots (4).$$

Equating these values of y,

$$\frac{7x-19}{3}=\frac{37-4x}{7}.$$

$$\therefore\ x=4.$$

Substituting this value of x in (3),

$$y=\frac{28-19}{3}=3.$$

Thus the solution is $x=4$, $y=3$.

III. Method of Cross Multiplication.

156. *This method consists in multiplying the given equations (reduced to the form* ax+by=c*) by such quantities as will render the coefficients of the same unknown numerically equal. By adding or subtracting the equations so found, we obtain a simple equation in one unknown.*

Examples.

(1.) Solve

$$7x-9y=5 \quad \dots\dots (1),$$
$$13x+4y=30 \quad \dots\dots (2).$$

Multiplying (1) by 4 and (2) by 9,

$$28x-36y=20 \quad \dots\dots (3),$$
$$117x+36y=270 \quad \dots\dots (4).$$

Adding (3) and (4),

$$145x=290,$$
$$\therefore\ x=2.$$

Again, multiplying (1) by 13 and (2) by 7,

$$91x-117y=65 \quad \dots\dots (5),$$
$$91x+28y=210 \quad \dots\dots (6).$$

Subtracting (5) from (6),

$$145y = 145,$$

$$\therefore\ y = 1.$$

Thus the solution is $x = 2,\ y = 1.$

(2.) Solve

$$8x + 25y = 9 \quad . \quad . \quad . \quad . \quad (1),$$

$$12x - 10y = 4 \quad . \quad . \quad . \quad . \quad (2).$$

Multiplying (1) by 2 and (2) by 5,

$$16x + 50y = 18 \quad . \quad . \quad . \quad . \quad (3),$$

$$60x - 50y = 20 \quad . \quad . \quad . \quad . \quad (4).$$

Adding (3) and (4),

$$76x = 38,$$

$$\therefore\ x = \tfrac{1}{2}.$$

Again, multiplying (1) by 3 and (2) by 2,

$$24x + 75y = 27 \quad . \quad . \quad . \quad . \quad (5),$$

$$24x - 20y = \ 8 \quad . \quad . \quad . \quad . \quad (6).$$

Subtracting (6) from (5),

$$95y = 19,$$

$$\therefore\ y = \tfrac{1}{5}.$$

Thus the solution is $x = \frac{1}{2},\ y = \frac{1}{5}.$

Exercise XLIX.

(1.) $4x + y = 11,\ x + 4y = 14.$

(2.) $2x + 3y = 21,\ 3x + 5y = 34.$

(3.) $3x = 23 - 2y,\ 10 + 2x = 5y.$

(4.) $\dfrac{x}{2} + \dfrac{y}{3} = 7,\ \ \dfrac{x}{3} + \dfrac{y}{2} = 8.$

(5.) $\dfrac{x}{5} + \dfrac{y}{6} = \dfrac{y}{2} + 2,\ \ \dfrac{x}{3} - 4 = \dfrac{3x}{10} - \dfrac{y}{4}.$

(6.) $3x - 2y = 3(6 - x),\ \ 3(4x - 3y) = 7y.$

(7.) $7(x - 1) = 3(y + 8),\ \ \dfrac{4x + 2}{9} = \dfrac{5y + 9}{2}.$

(8.) $\frac{2x-y}{4}-\frac{3}{2}=\frac{3y}{4}-x-2,\ x+y=8.$

(9.) $\frac{2y-6}{5}+\frac{5x-2}{2}=3x-2, \frac{x-5}{3}+\frac{5y-1}{2}=y+3.$

(10.) $2x-\frac{y-3}{5}=\frac{5x-2}{2},\ 2y-\frac{x-5}{3}=\frac{7y-7}{2}.$

(11.) $\frac{1}{10}(x+11)+\frac{1}{6}(y-4)=x-7, \frac{1}{7}(x+5)-\frac{1}{3}(y-7)=3y-x.$

(12.) $x-24=\frac{y}{2}+16,\ \frac{3}{5}(x+y)+x=\frac{3}{4}(2y-x)+105.$

(13.) $\frac{1}{3}(3x-7y)=\frac{1}{5}(2x+y+1),\ 8-\frac{1}{5}(x-y)=6.$

(14.) $\frac{2x+7y}{5}=1+\frac{2(2x-6y+1)}{3},\ \frac{x}{4}=y.$

(15.) $x+y=a,\ x-y=b.$

(16.) $ax+ay=a^2+b^2,\ x=a.$

(17.) $\frac{x}{a}+\frac{y}{b}=1,\ x+y=c.$

157. *When there are three simultaneous equations containing three unknowns, the solution is effected by eliminating one of the unknowns between the first and second equations, and also between the first and third, or second and third. Two equations are thus obtained involving two unknowns, which may be found by the methods already explained. The value of the third unknown may then be found by substitution.*

Example.

Solve
$$2x-3y+z=1 \quad \ldots \quad (1),$$
$$3x-5y+4z=3 \quad \ldots \quad (2),$$
$$4x+2y-3z=13 \quad \ldots \quad (3).$$

Multiplying (1) by 3 and (2) by 2,
$$6x-9y+3z=3 \quad \ldots \quad (4),$$
$$6x-10y+8z=6 \quad \ldots \quad (5).$$

Subtracting (5) from (4),

$$y-5z=-3 \quad . \quad . \quad . \quad . \quad (6).$$

Thus x is eliminated from (1) and (2).

Again, multiplying (1) by 2,

$$4x-6y+2z=2 \quad . \quad . \quad . \quad . \quad (7),$$

and $$4x+2y-3z=13 \quad . \quad . \quad . \quad . \quad (3).$$

Subtracting (7) from (3),

$$8y-5z=11 \quad . \quad . \quad . \quad . \quad (8).$$

And $$y-5z=-3 \quad . \quad . \quad . \quad . \quad (6).$$

From (6) and (8) we find $y=2$, $z=1$.

Substituting these values of y and z in (1), (2), or (3) we get $$x=3.$$

Thus the solution is $x=3$, $y=2$, $z=1$.

Exercise L.

(1.) $x+3y+2z=11$, $2x+y+3z=14$, $3x+2y+z=11$.

(2.) $x+2y+3z=13$, $2x+3y+z=13$, $3x+y+2z=10$.

(3.) $2x+3y-4z=10$, $3x-4y+2z=5$, $4x-2y+3z=21$.

(4.) $10x-2y+4z=10$, $3x+5y+3z=20$, $x+3y-2z=21$.

(5.) $3x+2y=13$, $3y+2z=8$, $3z+2x=9$.

(6.) $\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=3$, $4x+5y+6z=77$, $z+x=2y$.

(7.) $3x-2y=6$, $3y-2z=5$, $3z-2x=-2$.

(8.) $\frac{7}{3}(x-1)-y=35$, $9y-5z=43$, $x+y+z=30$.

(9.) $y-z+3=0$, $z-x=5$, $x+y=6$.

(10.) $y+z=a$, $z+x=b$, $x+y=c$.

CHAPTER XX.

PROBLEMS.

158. In the following problems the various unknowns are expressed in terms of separate and distinct symbols, and the relations between the quantities involved then take the form of equations. If two symbols x and y be employed, the conditions of the problem must give two independent equations; and three independent equations will be required to determine three unknowns, x, y, and z.

Examples.

(1.) A fraction becomes equal to 1 when 1 is added to the numerator, and equal to $\frac{1}{2}$ when 4 is added to the denominator. Find the fraction.

Let $\frac{x}{y}$ be the fraction.

Then by the conditions of the question,

$$\frac{x+1}{y}=1,$$

$$\frac{x}{y+4}=\tfrac{1}{2},$$

the solution of which is $x=5$, $y=6$.

Hence the fraction is $\frac{5}{6}$.

(2.) Find three numbers such that the sum of the first, one-fifth the second, and one-tenth the third, shall be equal

to 4; the sum of one-half the first, the second, and one-tenth the third equal to 7; and the sum of one-half the first, one-fifth the second, and the third equal to 12.

Let x= the first, y= the second, and z= the third number. Then by the conditions of the question,

$$x+\frac{y}{5}+\frac{z}{10}=4.$$

$$\frac{x}{2}+y+\frac{z}{10}=7.$$

$$\frac{x}{2}+\frac{y}{5}+z=12.$$

Multiply these severally by 10,

$$\begin{array}{lrl} & 10x+\ 2y+\ \ z=\ 40 & \ldots\ldots (1), \\ & 5x+10y+\ \ z=\ 70 & \ldots\ldots (2), \\ & 5x+\ 2y+10z=120 & \ldots\ldots (3), \\ 2\times(2)-(1), & 18y+z=100 & \ldots\ldots (4), \\ (3)-(2), & -8y+9z=\ 50 & \ldots\ldots (5), \\ 9\times(4)-(5), & 170y=850. & \\ & \therefore y=5. & \end{array}$$

Therefore from (4) $\quad z=100-18y=10$;

and from (1) $\quad x=2$.

The required numbers are thus 2, 5, and 10.

Exercise LI.

(1.) One of the digits of a number is greater by 5 than the other. When the digits are inverted, the number becomes $\frac{3}{8}$ of the original number. Find the digits.

(2.) In a division the majority was 162, which was $\frac{3}{11}$ of the whole number of votes; how many voted on each side?

(3.) The sum of two digits is 9. Six times one of the numbers they form is equal to 5 times the other number. Find the digits.

(4.) If the numerator and denominator of a fraction be each increased by 3, the fraction becomes 2; if each be increased by 11, it becomes $\frac{3}{2}$. Find the fraction.

(5.) A number consists of two digits whose sum is 12, and such that, if the digits be reversed in order, the number produced will be less by 36. Find the number.

(6.) Three towns A, B, and C are at the angles of a triangle. From A to C through B, the distance is 82 miles; from B to A through C, is 97 miles; and from C to B through A, is 89 miles. Find the direct distances between the towns.

(7.) The diameter of a five-franc piece is 37 *millimètres*, and of a two-franc piece is 27 *millimètres*. Thirty pieces laid in contact in a straight line measure one *mètre* exactly. How many of each kind are there?

(8.) At a contested election there are two members to be returned and three candidates, A, B, C. A obtains 2112 votes, B 1974, C 1866. Now 170 voted for B and C, 1500 for C and A, 316 for A and B. How many plumped for A, B, C, respectively?

(9.) A boat goes up stream 30 miles and down stream 44 miles in 10 hours. Again, it goes up stream 40 miles and down stream 55 miles in 13 hours. Find the rates of the stream and boat.

(10.) At a contested election there are two members to be returned, and three candidates, A, B, C. A obtains 1056 votes, B 987, and C 933. Now 85 voted for B and C, 744 for B only, 98 for C only. How many voted for C and A, how many for A and B, how many for A only?

(11.) Seventeen gold coins, all of equal value, and as many silver coins, all of equal value, are placed in a row at random. A is to have one-half of the row, B the other half. A's share is found to include seven gold coins, and the value of it is £6. The value of B's share is £6 15*s*. Find the value of each gold and silver coin.

(12.) The road from A to D passes through B and C successively. The distance between A and B is six miles greater than that between C and D, the distance between A and C is $\frac{1}{16}$ of a mile short of being half as great again as that between B and D, and the point half-way from A to D is between B and C half a mile from B. Determine the distances between A and B, B and C, C and D.

(13.) Fifteen octavos and twelve duodecimo volumes are arranged on a table, occupying the whole of it. After six of the octavos and four of the duodecimos are removed, only $\frac{5}{8}$ of the table is occupied. How many duodecimos only, or octavos only, might be arranged similarly on the table?

(14.) Three thalers are worth $\frac{1}{2}d.$ more than 11 francs. Five francs are worth $\frac{1}{2}d.$ more than 2 florins. One thaler is worth $2d.$ more than a franc and a florin together. Find the value of each coin in English money.

(15.) Six Prussian pounds weigh $\frac{1}{4}$ oz. more than 5 Austrian pounds. Twenty-five Austrian pounds weigh $\frac{1}{4}$ oz. more than 14 kilogrammes. One kilogramme weighs 1 oz. less than the sum of the weights of a Prussian and an Austrian pound. Find the number of ounces in each foreign measure of weight.

(16.) A person walks from A to B, a distance of $9\frac{1}{4}$ miles, in 2 hours and 52 minutes, and returns in 2 hours and 44 minutes, his rates of walking up hill, down hill, and on the level being 3, $3\frac{3}{4}$, and $3\frac{1}{4}$ miles an hour, respectively. Find the length of level ground between A and B.

CHAPTER XXI.*

EXPONENTIAL NOTATION.

159. Although the notation adopted in the preceding pages is sufficient for the purposes of the operations herein treated of, yet it is found expedient, before proceeding farther, to employ another notation to express roots, powers of roots, and their reciprocals. This notation, which consists in employing fractional exponents instead of radical signs and integral exponents, and negative exponents instead of reciprocal forms, possesses the great advantage of reducing to a few uniform laws the operations of Multiplication, Division, Involution and Evolution, with respect to powers, roots, and powers of roots, of a quantity, and their reciprocals.

160. The exponential notation consists in writing

$$a^{\frac{m}{n}} \text{ instead of } \sqrt[n]{a^m},$$

and

$$a^{-p} \quad \text{,,} \quad \text{,,} \quad \frac{1}{a^p}.$$

Thus, according to this notation,

$$a^{\frac{1}{2}}=\sqrt{a}, \qquad a^{\frac{2}{3}}=\sqrt[3]{a^2}, \qquad a^{\frac{5}{7}}=\sqrt[7]{a^5},$$

$$a^{-1}=\frac{1}{a}, \qquad a^{-3}=\frac{1}{a^3}, \qquad a^{-8}=\frac{1}{a^8},$$

$$a^{-\frac{1}{2}}=\frac{1}{a^{\frac{1}{2}}}=\frac{1}{\sqrt{a}}, \qquad a^{-\frac{3}{4}}=\frac{1}{a^{\frac{3}{4}}}=\frac{1}{\sqrt[4]{a^3}},$$

* This Chapter may be omitted by those who do not intend to read more advanced works on Algebra.

$$2a^{-3}=\frac{2}{a^3},\qquad 5a^{-\frac{1}{2}}=\frac{5}{\sqrt{a}},\qquad 4a^{-\frac{3}{5}}=\frac{4}{\sqrt[5]{a^3}}.$$

161. When the exponential notation is employed, the quantity is said to be raised to the *power* indicated by the exponent.

Thus	$a^{\frac{1}{2}}$ is read	a	*to the*	*power*	$\frac{1}{2}$;
	$a^{\frac{3}{4}}$ „	a	„	„	$\frac{3}{4}$;
	x^{-3} „	x	„	„	-3;
	$x^{-\frac{4}{5}}$ „	x	„	„	$-\frac{4}{5}$.

The term *power* in this extended Algebraical sense thus includes the terms *power, root, root of a power, reciprocal of a power, reciprocal of a root, reciprocal of a root of a power*, as used in the ordinary or Arithmetical sense.

Exercise LII.

Express in the Arithmetical notation—

(1.) $a^{\frac{1}{3}},\ a^{\frac{1}{6}},\ a^{\frac{2}{5}},\ a^{\frac{3}{2}}$.

(2.) $x^{-2},\ x^{-8},\ x^{-10}$.

(3.) $m^{-\frac{1}{3}},\ n^{-\frac{5}{2}},\ p^{-\frac{7}{4}}$.

(4.) $2a^{\frac{1}{4}},\ 3x^{-2},\ 6m^{-\frac{3}{2}}$.

Express in the Exponential notation—

(5.) $\sqrt{x},\ \sqrt[5]{m},\ \sqrt[8]{n}$.

(6.) $\frac{1}{x},\ \frac{1}{a^2},\ \frac{1}{a^5},\ \frac{1}{a^8}$.

(7.) $\sqrt{x^3},\ \sqrt[3]{x^2},\ \sqrt[5]{x^4}$.

(8.) $\frac{1}{\sqrt{x}},\ \frac{1}{\sqrt[3]{x}},\ \frac{1}{\sqrt[5]{x}}$.

(9.) $\frac{2}{m}$, $\frac{3}{n^2}$, $\frac{10}{p^3}$.

(10.) $\frac{2}{\sqrt{x}}$, $\frac{5}{\sqrt[3]{x}}$, $\frac{7}{\sqrt[5]{x^2}}$.

162. The utility of employing the exponential notation will be exhibited in the statement of the three following rules, which are usually called *Index Laws.*

I. *The product of any two powers of the same quantity is a power whose exponent is the Algebraic sum of the exponents of the factors.*

Since the product of a quantity and its reciprocal$=1$, this rule cannot be applied when the exponents of the two factors are numerically equal and of opposite signs, *unless the zero power of a quantity be considered*$=1$.

Examples.

(1.) $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}=a^{\frac{1}{2}+\frac{1}{3}}=a^{\frac{5}{6}}$.

(2.) $a^{\frac{3}{4}} \cdot a^{\frac{5}{6}}=a^{\frac{3}{4}+\frac{5}{6}}=a^{\frac{19}{12}}$.

(3.) $a^3 \cdot a^{-1}=a^{3-1}=a^2$.

(4.) $a^{-2} \cdot a^{-3}=a^{-2-3}=a^{-5}$.

(5.) $a \cdot a^{-\frac{1}{2}}=a^{1-\frac{1}{2}}=a^{\frac{1}{2}}$.

(6.) $a^{\frac{5}{6}} \cdot a^{-\frac{3}{4}}=a^{\frac{5}{6}-\frac{3}{4}}=a^{\frac{1}{12}}$.

(7.) $a \cdot a^{-1}=a^{1-1}=a^0=1$.

(8.) $a^{\frac{1}{2}} \cdot a^{-\frac{1}{2}}=a^0=1$.

This law may also be expressed briefly as follows:—

$$a^m \cdot a^n=a^{m+n}$$

where m *and* n *are any quantities whatsoever, positive or negative, integral or fractional, including zero if* $a^0=1$.

163. Proofs of the preceding rule will be exhibited in the following particular cases.

(1.) If m and n be positive integers,

$$a^m \cdot a^n = aaa \ldots \text{ (}m\text{ factors)} \times aaa \ldots \text{ (}n\text{ factors)},$$
$$= aaa \ldots \text{ (}m+n\text{ factors)},$$
$$= a^{m+n}.$$

(2.) $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}} = \sqrt{a}\ \sqrt[3]{a} = \sqrt[6]{a^3}\ \sqrt[6]{a^2} = \sqrt[6]{a^5} = a^{\frac{5}{6}} = a^{\frac{1}{2}+\frac{1}{3}}.$

Here $\sqrt{a} = \sqrt[6]{a^3}$, because each of them when multiplied by itself six times produces a^3. So also $\sqrt[3]{a} = \sqrt[6]{a^2}$; and generally, if $m, n, p,$ are integers,

$$\sqrt[np]{a^{mp}} = \sqrt[n]{a^m}$$

because each of these quantities when multiplied by itself np times produces a^{mp}.

(3.) If m, n, p, q be positive integers, then

$$a^{\frac{m}{n}} \cdot a^{\frac{p}{q}} = \sqrt[n]{a^m} \cdot \sqrt[q]{a^p} = \sqrt[nq]{a^{mq}} \cdot \sqrt[nq]{a^{pn}}.$$
$$= \sqrt[nq]{a^{mq} \cdot a^{pn}}$$
$$= \sqrt[nq]{a^{mq+pn}}, \quad \text{by Ex. 1.}$$
$$= a^{\frac{mq+pn}{nq}}$$
$$= a^{\frac{m}{n}+\frac{p}{q}}.$$

(4.) $a^2 \cdot a^{-3} = \dfrac{a^2}{a^3} = \dfrac{1}{a} = a^{-1} = a^{2-3}.$

Exercise LIII.

Find the products of

(1.) $2x,\ 3x^n;\ x^2,\ 4x^m;\ 4x^m,\ x^{2m}.$

(2.) $x^{\frac{1}{2}},\ 2x^{\frac{1}{4}};\ 3x^{\frac{1}{3}},\ 2x^{\frac{1}{2}};\ 6x^{\frac{3}{4}},\ 5x^{\frac{1}{2}}.$

(3.) $x^n,\ x^{\frac{1}{2}};\ 2x^{\frac{n}{2}},\ 3x^n;\ x^{\frac{n}{2}},\ x^{\frac{n}{3}}.$

(4.) $2a^3,\ a^{-2};\ a^{-4},\ 3a^3;\ 5a,\ 6a^{-2}$.

(5.) $a^{\frac{1}{2}},\ a^{-\frac{1}{3}};\ 2a^{\frac{3}{4}},\ a^{-\frac{5}{6}};\ a,\ a^{-\frac{1}{3}}$.

(6.) $a^{\frac{p}{2}},\ a^{-\frac{p}{3}};\ a^n,\ a^{-\frac{n}{2}};\ a^{2n},\ a^{-\frac{n}{3}}$.

(7.) $a^3,\ a^{-3};\ a^n,\ a^{-n};\ 2a,\ 3a^{-1};\ ma^2,\ na^{-2}$.

164. II. *When one power of a quantity is divided by another, the quotient is a power whose exponent is the Algebraic difference between the exponents of the dividend and divisor.*

Since the quotient is $=1$ when the dividend and divisor are the same, this rule cannot be applied in the case of equal powers *unless the zero power of a quantity be considered* $=1$.

Examples.

(1.) $\frac{a^5}{a^3}=a^{5-3}=a^2$.

(2.) $\frac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}}=a^{\frac{1}{2}-\frac{1}{3}}=a^{\frac{1}{6}}$.

(3.) $\frac{a^2}{a^3}=a^{2-3}=a^{-1}$.

(4.) $\frac{a^3}{a^{-4}}=a^{3+4}=a^7$.

(5.) $\frac{a^{-\frac{3}{4}}}{a^{-\frac{5}{6}}}=a^{-\frac{3}{4}+\frac{5}{6}}=a^{\frac{1}{12}}$.

(6.) $\frac{a^5}{a^5}=a^{5-5}=a^0=1$.

(7.) $\frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}}=a^{\frac{1}{2}-\frac{1}{2}}=a^0=1$.

This rule may also be expressed briefly as follows :—

$$\frac{a^m}{a^n}=a^{m-n}$$

where m *and* n *are any quantities whatsoever, positive or negative, integral or fractional, including zero if* $a^0=1$.

165. The proof of this rule rests on that of the preceding. For since $a^{m-n}.a^n=a^m$, it follows that

$$a^{m-n}=\frac{a^m}{a^n}.$$

Also, since $\frac{1}{a^n}=a^{-n}$, the second rule may be considered to be included under the first. Thus

$$\frac{a^m}{a^n}=a^m.\ a^{-n}=a^{m-n}.$$

EXERCISE LIV.

Divide

(1.) a^{2m} by a^m; a^{3n} by a^n.

(2.) $a^{\frac{2}{3}}$ by $a^{\frac{1}{3}}$; a by $a^{\frac{2}{3}}$.

(3.) $x^{\frac{3}{2}}$ by x; $x^{\frac{5}{4}}$ by $x^{\frac{1}{4}}$.

(4.) x^2 by x^{-1}; x^3 by x^{-2}.

(5.) x^{-1} by x^{-2}; x^{-2} by x^{-5}.

(6.) $x^{\frac{2}{3}}$ by $x^{-\frac{1}{3}}$; $x^{\frac{3}{2}}$ by x^{-1}.

(7.) x^{-n} by x^{-2n}; $x^{\frac{n}{2}}$ by x^{-n}.

166. III. *The power of a power of a quantity is a power whose exponent is the product of the numbers expressing those powers.* In other words,

$$(a^m)^n=a^{mn}$$

where m *and* n *are any quantities whatsoever, positive or negative, integral or fractional, including zero if* $a^0=1$.

Examples.

(1.) $(a^2)^3=a^6$; $(a^5)^2=a^{10}$.

(2.) $(a^{\frac{1}{3}})^2=a^{\frac{2}{3}}$; $(a^{\frac{2}{3}})^6=a^4$.

(3.) $(a^{\frac{1}{2}})^{\frac{1}{3}}=a^{\frac{1}{6}}$; $(a^{\frac{2}{3}})^{\frac{1}{2}}=a^{\frac{1}{3}}$.

(4.) $(a^{-1})^2=a^{-2}$; $(a^{-2})^3=a^{-6}$.

(5.) $(a^{-1})^{-2}=a^2$; $(a^{-3})^{-4}=a^{12}$.

(6.) $(a^{\frac{1}{3}})^{-2}=a^{-\frac{2}{3}}$; $(a^{\frac{1}{3}})^{-\frac{3}{2}}=a^{-\frac{1}{2}}$.

167. Proofs of the preceding rule will be exhibited in the following particular cases:—

(1.) $(a^2)^3=a^2 \cdot a^2 \cdot a^2=a^6=a^{3\times 2}$.

(2.) $(a^{\frac{1}{3}})^2=a^{\frac{1}{3}} \cdot a^{\frac{1}{3}}=a^{\frac{1}{3}+\frac{1}{3}}=a^{\frac{2}{3}}$

(3.) If n be a positive integer,

$$(a^m)^n=a^m \cdot a^m \quad . \quad . \text{ to } n \text{ factors,}$$
$$=a^{m+m+ \;....\text{ to } n \text{ terms}},$$
$$=a^{mn}.$$

(4.) If p and q be positive integers,

$$(a^n)^{\frac{p}{q}}=a^{\frac{np}{q}},$$

because each of these quantities when raised to the qth power produces a^{np}.

EXERCISE LV.

Express the following as powers of a:—

(1.) $(a^4)^3$; $(a^2)^4$; $(a^3)^3$.

(2.) $(a^{-1})^2$; $(a^{-2})^3$; $(a^{-3})^4$.

(3.) $(a^2)^{-3}$; $(a^{-2})^{-3}$; $(a^{-3})^{-4}$.

(4.) $(a^{\frac{1}{2}})^5$; $(a^{\frac{3}{2}})^6$; $(a^{\frac{5}{2}})^4$.

(5.) $(a^{\frac{2}{3}})^{\frac{3}{2}}$; $(a^{\frac{3}{4}})^{\frac{2}{3}}$; $(a^{\frac{5}{4}})^{\frac{4}{3}}$.

(6.) $(a^{-\frac{2}{3}})^{-1}$; $(a^{-\frac{2}{3}})^{-2}$; $(a^{-\frac{4}{5}})^{-\frac{5}{6}}$.

168. The Index Laws (I., II., III.) are thus seen to be true on the assumption that

$$a^{\frac{m}{n}}=\sqrt[n]{a^m}, \quad a^{-p}=\frac{1}{a^p}, \quad a^0=1,$$

for all values of m and n, positive or negative, integral or fractional, including zero.

Instead, however, of treating the subject of the Index Laws and notation as in the preceding Articles, we may proceed as follows:—

If m and n be positive integers, it may be proved, as is done in Arts. 163, 165, 167, that

I. $a^m \cdot a^n = a^{m+n}$.

II. $\frac{a^m}{a^n} = a^{m-n}$, m being greater than n.

III. $(a^m)^n = a^{mn}$.

These laws, which are thus *proved* to hold in the particular case where m and n are positive integers, are then *assumed* to be true when m and n are any quantities whatsoever, positive or negative, integral or fractional, including zero; and from this extension of these laws we deduce that

$a^{\frac{m}{n}}$ must$=\sqrt[n]{a^m}$, $a^{-p} = \frac{1}{a^p}$, and $a^0 = 1$.

Thus, by I., $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a$.

But $\sqrt{a} \cdot \sqrt{a} = a$.

$\therefore a^{\frac{1}{2}} = \sqrt{a}$.

By III., $(a^{\frac{3}{4}})^4 = a^3$.

But $(\sqrt[4]{a^3})^4 = a^3$.

$\therefore a^{\frac{3}{4}} = \sqrt[4]{a^3}$.

By I., $a^3 \cdot a^0 = a^3$.

But $a^3 \times 1 = a^3$.

$\therefore a^0 = 1$.

By I., $a^3 \cdot a^{-3} = a^0 = 1$.

But $a^3 \times \frac{1}{a^3} = 1$.

$\therefore a^{-3} = \frac{1}{a^3}$.

169. From the remarks of this chapter it will be thus seen that, although there is no absolute necessity for using such symbols as $a^{\frac{3}{4}}$, a^{-4}, $a^{-\frac{2}{3}}$, still their introduction gives rise to a uniformity in certain Algebraical processes; and the number of rules which otherwise would be required to meet the different cases that arise in these operations thus becomes largely reduced.

ANSWERS.

I.

(1.) $56+10+15$; $10+12+\frac{4}{5}$; $2+\frac{3}{4}+\frac{6}{7}$; $\frac{5}{6}+\frac{9}{10}+\frac{11}{12}+\frac{1}{3}$.
(2.) $29-15$; $2\frac{1}{3}-1\frac{1}{2}$; $2{\cdot}5-1{\cdot}6$.
(3.) $11+35+6-17$; $81+75-69-42$.
(4.) $15-7+8+9$.
(5.) $28-16+10-4$.

II.

(1.) A's -60, $+20$; B's $+60$, -30; C's $+30$, -20.
(2.) A's $+20$, $+20$, -30, -40; B's -20, $+30$, $+30$, -40; C's -20, -30, $+40$, $+40$.
(3.) A -10, $+4$; B -7, $+10$; C -4, $+7$.
(4.) $-2°$, $+5°$, $-3°$.
(5.) $+1°$, $-1°$, $+1°$, $-1°$, $+1°$.
(6.) $-2°$, $+2°$, $-2°$, $+2°$.
(7.) $+25°$, $-7°$.

III.

(1.) 30.	(2.) 20.	(3.) 22.
(4.) 22.	(5.) 24, 120, 0.	(6.) 14.
(7.) 7.	(8.) 16.	(9.) $\frac{13}{10}$.

(10.) $\frac{1}{4}$. (11.) $\frac{23}{30}$. (12.) $\frac{2039}{1080}$.
(13.) $2a^3$, $3a^3$, $2a^2+3a^3+4a^4$.

IV.

(1.) $2, 3a, \frac{4}{5}, bc, 4a^2b, \frac{1}{2}a$.
(2.) $-1, +3, -\frac{5}{7}, +2b, -5x^2$.
(3.) $+1, -1, -3a, +\frac{2}{3}cd$.
(4.) $-1, +3a^2, -\frac{2}{3}az$. (5.) $\frac{1}{2}, \frac{2}{5}, -\frac{1}{4}, +\frac{3}{7}, -\frac{5}{6}$.
(6.) $1, -1, -2, -3, +\frac{1}{2}, -\frac{3}{4}$. (7.) $2x, x$.
(8.) $-a^2, +\frac{2}{3}a^2; +2a^2x, -a^2x$.
(9.) $-3a^2x, +2a^2x; ax^2, -ax^2$.

V.

(1.) $+25$. (2.) $+4\frac{1}{12}$. (3.) -28.
(4.) $-2\frac{11}{20}$. (5.) $+5$. (6.) -6.
(7.) $+\frac{1}{12}$. (8.) $-\frac{5}{36}$. (9.) $-0{\cdot}7$.
(10.) $+11$. (11.) $+2$. (12.) -12.
(13.) $-\frac{17}{28}$. (14.) $+1\frac{7}{15}$. (15.) $+0{\cdot}342$.

VI.

(1.) $+3a$. (2.) $+15a$. (3.) $-5a$.
(4.) $-13a$. (5.) $+2x^2$. (6.) $+3a^2$.
(7.) $-3a$. (8.) $+10c$. (9.) $-5c$.
(10.) $-4x^2$. (11.) $+7ab$. (12.) $+\frac{1}{6}a$.
(13.) $-\frac{2}{35}a^2$. (14.) $-\frac{8}{3}a$. (15.) $-\frac{22}{15}a$.

VII.

(1.) $2a-3b$. (2.) $-x+3y$. (3.) $-2x-3y-z$.
(4.) $3a+2x-5y$. (5.) $4-x+2y$.
(6.) $a^2-b^2+\frac{1}{2}$. (7.) $a-2b+3c-d$.
(8.) $-x+2y-z+1$. (9.) $4a-2b$.

(10.) a^2-4bc. (11.) $4a-9b$.
(12.) $-2a-b-c$. (13.) $2x-6y-z$.
(14.) $4a+x+5$. (15.) $5a-4x$.
(16.) $2a+2b+2c$. (17.) $10x+3y-z$.
(18.) $4x^2$. (19.) $4a^3+4b^3$. (20.) $a+b$.
(21.) $\frac{x}{4}+\frac{5}{12}y+\frac{5}{12}z$. (22.) $2a+\frac{3}{2}b-\frac{5}{3}c$.

VIII.

(1.) 4. (2.) 2. (3.) -9.
(4.) -3. (5.) 5. (6.) -3.
(7.) 10·6. (8.) $-3·39$. (9.) $-2·16$.
(10.) $-0·966$. (11.) $-a$. (12.) $-7x$.
(13.) $10a^2$. (14.) $2c$. (15.) $2a-\frac{b}{2}$.
(16.) $-3a^2$. (17.) $11x^2$. (18.) $3a+b$.
(19.) $2x$. (20.) $3a-x+4$. (21.) $-5ab+2b^2$.
(22.) $-ax-5by+4cz$. (23.) $-6a+b+x-4$.
(24.) $-7x^3+28x^2-5x$. (25.) $\frac{x}{4}+\frac{y}{2}+\frac{3z}{2}$.
(26.) $\frac{1}{2}a+\frac{1}{2}b-\frac{1}{6}c$.

IX.

(1.) $2x^2+-1$; $3x+-4y+-5$; $2a+-3b++4c$.
(2.) $2a--5a$. (3.) $-6-+5x$.
(4.) $2a+-3b-+7$. (5.) $5++x--3a$.
(6.) $5a+(b-4)$. (7.) $-a+(-b+5)$.
(8.) $a-4+(2b-c)$. (9.) $x^2+(2y+5)-z$.
(10.) $a-1+(3b+5)--3c$.
(11.) $x+(2x^2-1)+(-3x^2-8)$. (12.) $4a^2-(b^2-c)$.
(13.) $a^2+4-(-2b+3)$. (14.) $2a-5-(a^2-2a+3)$.
(15.) $a+b+c+(a-b-c)-(-a+2b-3c)$.

X.

(1.) $2a+3b-c$. (2.) $ab-bc+c$.
(3.) x^2-4x-1. (4.) $5x^3-3x^2+7x-8$.
(5.) $8a-b-c$. (6.) $8a-b+c$.
(7.) $8a+2b-3c$. (8.) $2a-6+b+c$.
(9.) $x+11+4y$. (10.) b. (11.) $2x^2+3x$.

XI.

(1.) $(a-1)(2a^2-3)$; $(-2+a)(-3-a^2)$; $(x-5)(-2x+7)$.
(2.) $-2a^2(b^2-1)$; $(a^2-1)(-3a)$; $5x(-x^2+3)$.
(3.) $\frac{1}{3}(x-1)$; $\frac{1}{5}(2x-3)$; $-\frac{2}{3}(x^2-5)$.
(4.) $-5x(x-1)(x+2)$; $(x^2-4)(+5x)(2a+3)$.
(5.) $+8x(-5y)(xy-1)$; $-7a(ab-3)(+8b)$.

XII.

(1.) $-6ab$; $-5ac$; $+6a^2b$; $-30xy$.
(2.) $-14abc^2$; $-20a^2bc$; $-16xyz$; $+48a$.
(3.) $-\frac{3}{2}xy$; $-\frac{3}{2}ab$; $+\frac{1}{6}xy$; $-\frac{2}{5}a^2b$.
(4.) $-6x^3y^3$; $+3ax^3y^5$; $-\frac{3}{5}a^3b^3c^4$.
(5.) $-\frac{a^3}{3}$; $\frac{a^3b^4}{15}$; $-\frac{3x^3y^5}{20}$; $+\frac{abx^3y^2}{12}$.

XIII.

(1.) $-8ax+6bx-2cx$; $12x^2y-8xy+4y$; $-2abd+3cd$.
(2.) $3x^3-6x^2-15x$; $-2a^5+3a^4-7a^3$; $-4ax^3+4a^2x^2-8a^3x$.
(3.) $4x^2yz+2xy^2z-6xyz^2$; $-28a^3b^2+4a^2b^3-4a^2b^2$.
(4.) $2a^2b+ab^2-\frac{6}{5}abc$; $-a^3+\frac{3}{2}a^2-2a$; $\frac{1}{3}ax^3-\frac{1}{2}a^2x^2+\frac{5}{8}a^3x$.
(5.) $-12a+10ab-15$; $\frac{4}{15}ax^2-\frac{14}{15}ax+\frac{7}{5}a^2x^2$.
(6.) $5x^2-5x+20$; $-2a+2ab-6$; $-a^3+2a^2x-a$.
(7.) $ax-x^2$; $ac-bc+c^2$; $-5ab+5a^2b^2-15a^3b$.
(8.) $4a^3-6a^2+8a$; $+2x^2-\frac{2}{5}x^3+3x^5$.

XIV.

(1.) $2x^2+5x-12$; $-4x^2-x+5$; $2-x-3x^2$.

(2.) $2x^3-x^2-4x+2$; $2-2x+3x^2-3x^3$; $3+3x^2-x^3-x^5$.

(3.) $6a^3-7a^2+14a-8$; $1+a^3$; $1-a^3$.

(4.) $am-an+bm-bn$; $am+bm-cm+2an+2bn-2cn$; $4m^2-n^2$.

(5.) $x^2y^2-x^4$; $x^3+x^3y-x^2y^2-2xy-2xy^2+2y^3$.

(6.) $2a^4+a^3-22a^2+23a-4$; $a^2-4b^2+12bc-9c^2$.

(7.) $3a^2-4ab+8ac-4b^2+8bc-3c^2$; $3x^4-4x^3y+6x^2y^2+4xy^3+3y^4$.

(8.) $x^4-\frac{5}{6}x^2+\frac{1}{6}$; $a^2+\frac{1}{6}a-\frac{1}{6}$; $2a^2-\frac{1}{8}$.

(9.) $2x^3-\frac{6}{5}x^2+\frac{21}{10}x-\frac{1}{5}$; $9x^3-\frac{7}{2}x^2+\frac{5}{6}x-\frac{1}{12}$.

(10.) $\frac{x^2}{4}-\frac{23xy}{8}-\frac{x}{2}-\frac{3y^2}{2}+6y$; $x^3-\frac{56}{9}x^2+\frac{25}{12}x-\frac{1}{6}$.

(11.) $\frac{1}{4}a^4-\frac{4}{9}a^2+\frac{4}{3}a-1$. (12.) $x^3+3xy-2x+y^3-2y+1$.

(13.) $a^3-3abc+b^3+c^3$.

XV.

(1.) $+30a^5b^3$; $-\frac{1}{15}x^7$; $+\frac{16}{3}x^3y^4z^2$.

(2.) $-24x^4y+30x^2y^2$; $-2a^5b+3a^4b^2-a^3b$.

(3.) $8x^3-26x^2-17x+6$. (4.) x^4-x^3+x-1.

(5.) $x^8-2a^4x^4+a^8$.

XVI.

(1.) $(2a-5)\div-3a$; $(4a^2-3a+1)\div(3a-4)$.

(2.) $2a\div-3b$; $-x^2\div+2x$; $3x\div2a$.

(3.) $4x^2\div(2x-5)$; $-ax^2\div(x-a)$.

XVII.

(1.) -4; -5; $+\frac{4}{5}$; $-\frac{40}{3}$. (2.) $-\frac{a}{2x}$; $-\frac{3a^2}{2b}$; $+\frac{2xy}{a}$.

(3.) $-\frac{3x}{2y}$; $-\frac{6ax}{5b}$; $+\frac{9x^2}{14a^2}$. (4.) $2a^3$; a^3; $4x^2$.

(5.) $-2a$; $5ab^2c^3$. (6.) $-\frac{1}{3}xy^3$; $+\frac{3}{5}ay$.

XVIII.

(1.) $-2a+3b-4$; $ax-3+2a^2$.

(2.) $-4x+3-a$; $4x^2-2x+3$.

(3.) $-a^2+4a-5$; $x^2-3xy+4y^2$.

(4.) $2a-3b+c$. (5.) $-4ac^2+3bc^3-1$.

(6.) $(a-b)x$, $(2a-c+1)x$. (7.) $(4-a)xy$, $(3x-y)xy$.

XIX.

(1.) $x-4$; $3x+1$. (2.) $x+1$; $2x-3$.

(3.) $3x+2$; $3x^2-2x+6$.

(4.) $x-1$; x^2-x+1. (5.) x^2-3x+1.

(6.) x^3+x^2+x+1; $x^4-x^3+x^2-x+1$.

(7.) $x-y$; x^2-xy+y^2. (8.) a^2+ab+b^2.

(9.) $5x+6y-3$. (10.) x^2-ax+a^2.

(11.) $x^2-2xy+y^2$. (12.) $\frac{3}{5}ax-2x^2$.

XX.

(1.) $2x^2-3x+\frac{11}{2}$, $-\frac{7}{2}$. (2.) $x-a$, $2a^2$.

(3.) x^2-ax+a^2, $-2a^3$. (4.) x^2-x+4, $-3x-4$.

(5.) $2x^2+3$, $-5x^2-3x-3$.

XXI.

(1.) $3x-a$ miles. (2.) $50+x$ dollars.

(3.) $2b$ miles. (4.) x^2-y^2 square feet.

(5.) $\frac{x}{a}$ hours. (6.) $x+\frac{y}{2}+\frac{3z}{4}$.

(7.) $x+\frac{y}{20}+\frac{z}{240}$. (8.) $\frac{10x}{a}$.

(9.) $\frac{ax}{9}$. (10.) $\frac{x^4}{100}-\frac{y^4}{100}$.

(11.) $\frac{3a}{2}+\frac{4b}{3}-5$. (12.) $3b$ acres.

(13.) $\frac{15a}{t}$ miles. (14.) $\frac{ax}{10}$ hours.

(15.) $6ax+a$ dollars.

XXII.

(1.) 2. (2.) 10. (3.) 7.
(4.) -5. (5.) 2. (6.) 5.
(7.) $-\frac{1}{3}$. (8.) -5. (9.) $\frac{3}{2}$.
(10.) 2. (11.) 3. (12.) 2.
(13.) 7. (14.) 10. (15.) $2\frac{1}{7}$.
(16.) $\frac{1}{2}$. (17.) 6. (18.) $6\frac{1}{2}$
(19.) 7. (20.) 13. (21.) a.
(22.) $3a$. (23.) $a+b$. (24.) $2b$.
(25.) $b+c$. (26.) 1. (27.) a^2+ab+b^2.

(28.) $\frac{ab-ac}{a-c}$.

XXIII.

(1.) 12. (2.) 12. (3.) 24.
(4.) 30. (5.) $23\frac{1}{4}$. (6.) $6\frac{6}{13}$.
(7.) 6. (8.) 3. (9.) 3.
(10.) 9. (11.) 4. (12.) 5.
(13.) $3\frac{1}{2}$. (14.) -2. (15.) 33.
(16.) $2\frac{1}{2}$. (17.) 2. (18.) 2.
(19.) 120. (20.) 13.

XXIV.

(1.) 15 and 10. (2.) 60 and 75. (3.) 20 and 17.
(4.) 14 lbs. (5.) 26, 17. (6.) 181 and 145.

(7.) 5. (8.) 14 years. (9.) In 9 years.
(10.) 18. (11.) 66 years. (12.) 400.
(13.) 700. (14.) 30 for translation, 5 for mathematics. 4 for Latin prose.
(15.) 21 shillings. (16.) A's £300, B's £100.
(17.) 400 inches. (18.) 18, 11 and 8.
(19.) 35. (20.) 200 quarters. (21.) 12 lbs.
(22.) 150 lbs. (23.) 240 sovereigns, 480 shillings, 720 pence. (24.) 13.
(25.) £3000 at 5 per cent., £10,000 at 4 per cent. (26.) £450 at $4\frac{1}{2}$ per cent. £350 at $5\frac{1}{2}$ per cent.
(27.) 1800 infantry, 600 artillery, 200 cavalry. (28.) 17 years.
(29.) 26. (30.) 12.
(31.) 56 workmen; 150 shillings. (32.) 1330.
(33.) 4290 feet. (34.) 30,000 men.
(35.) 200 miles from Edinburgh. (36.) In 56 hours.

XXV.

(1.) x^2-2x+1, $x^2+2ax+a^2$, $x^2-10x+25$, x^2+6x+9.

(2.) $4x^2+4x+1$, $9x^2-6x+1$, $4x^2+12x+9$, $9x^2-12x+4$.

(3.) $x^4-2ax^2+a^2$, $4x^2y^2+4xy+1$, $9x^4-12ax^2+4a^2$, $a^2x^4-8abx^2+16b^2$.

(4.) $x^2+y^2+z^2-2xy+2xz-2yz$, $4x^2+9y^2+z^2+12xy-4xz-6yz$, $x^2+4y^2+25z^2-4xy-10xz+20yz$, $4x^2+16y^2+1-16xy+4x-8y$.

(5.) $4a^4+4a^3+13a^2+6a+9$, $9a^4-24a^3+22a^2-8a+1$, $a^4-4a^3-4a^2+16a+16$.

(6.) 2401, 9604, 990025.

XXVI.

(1.) x^2-1; a^2-9; $4-x^2$.

(2.) $4x^2-1$; $25a^2-4$; $16x^2-a^2$.

(3.) a^4-x^2; a^6-1; $a^{10}-x^4$.

(4.) $9a^4-4b^2$; $16a^6-4x^4$; $49a^8-25a^6$.

(5.) 2496; 9975; 489975.

XXVII.

(1.) m^3+n^3; p^3-q^3.

(2.) m^3+1; $1-q^3$.

(3.) x^3+27; a^3-64.

(4.) $8a^3+1$; $64x^3-a^3$.

(5.) $8a^3+27b^3$; $27x^3-125y^3$.

(6.) x^6-1; x^9+a^6.

XXVIII.

(1.) x^2-x+1; $x^4-x^3+x^2-x+1$.

(2.) x^2+x+1; $x^4+x^3+x^2+x+1$.

(3.) $x-1$; x^3-x^2+x-1.

(4.) $x+1$; x^3+x^2+x+1.

(5.) $2a-3b$.

(6.) $3x^3+2a$.

(7.) $\frac{1}{2}a^2-x^3$.

(8.) $4a^2-6ab+9b^2$; a^4-2a^2+4.

(9.) $9a^2+3ab+b^2$; $4a^4+6a^2b+9b^2$.

XXIX.

(1.) $(-a)^3$, $(2x)^3$, $(3xy^2)^3$, $(2a^4b^2c)^3$.

(2.) $(2a-1)^2$, $(a-b+1)^2$, $(x^3-1)^2$.

(3.) $\{(x^3)^4\}^2$, $\{(-2a)^3\}^2$, $\{(4ax)^5\}^2$, $\{(3a^2bc^4)^3\}^2$.

(4.) $\{(a-b)^2\}^3$, $\{(x^2-1)^4\}^3$, $\{(x^2-3x+2)^2\}^3$.

(5.) $(x^3)^2$, $\{(-2x)^3\}^2$, $\{(a^2b)^3\}^2$, $\{(x-a)^3\}^2$, $\{(x^2-ax+1)^3\}^2$.

(6.) $\{(-a)^2\}^3$, $\{(x^2)^2\}^3$, $\{(4x-1)^2\}^3$, $\{(x^3-a^3)^2\}^3$.

XXX.

(1.) x^6, $8x^6$, x^9, $8x^3$, $81x^8$.

(2.) a^2x^4, a^4x^6, $a^4x^6y^3$.

(3.) $a^3b^3c^6$, $a^6b^3c^6$, $8a^6b^9c^{12}$.

(4.) $x^{10}y^{15}z^{20}$, $a^6b^{24}c^{42}$.

(5.) x^2+2x+1, $4x^2-12x+9$, x^4-10x^2+25, $x^6-4a^2x^3+4a^4$.

(6.) $x^4+4x^3+10x^2+12x+9$, $x^4-6x^3+17x^2-24x+16$, $4x^6-4x^5+x^4+20x^3-10x^2+25$.

XXXI.

(1.) $\sqrt{2x}$, $\sqrt{ax^2}$, $\sqrt{x^2-1}$, $\sqrt{x^2-3x+4}$.

(2.) $\sqrt[3]{-x^6}$, $\sqrt[3]{3a^3}$, $\sqrt[3]{a-b}$, $\sqrt[3]{(a^3-3a+4)}$.

(3.) $\sqrt{\sqrt[3]{3ax}}$, $\sqrt{\sqrt[3]{x-1}}$, $\sqrt{\sqrt[3]{x^6+1}}$.

(4.) $\sqrt[4]{\sqrt[3]{2}}$, $\sqrt[4]{\sqrt[3]{3x-1}}$, $\sqrt[4]{\sqrt[3]{2a^4-a^2+3}}$.

(5.) $\sqrt[4]{a}$, $\sqrt[6]{2x}$, $\sqrt[6]{3x}$, $\sqrt[12]{2a}$.

(6.) $\sqrt[6]{x^4-1}$, $\sqrt[12]{2x^3-5}$, $\sqrt[20]{x^8-6x^4+7}$.

XXXII.

(1.) $2a^2b$, $5xy^3$, $9x^2y^4$. (2.) $4x+5$.

(3.) $6x-3$. (4.) $1+3x$. (5.) $x+\frac{1}{4}$.

(6.) $x-\frac{7}{2}$. (7.) $2x-\frac{1}{12}$. (8.) $2x-3y$.

(9.) x^2+2x+1. (10.) x^2+x+1.

(11.) $x^2-2xy+y^2$. (12.) $2x^3-x^2-3x+2$.

(13.) $1+\dfrac{x}{2}-\dfrac{x^2}{8}$; remainder $\dfrac{x^3}{8}-\dfrac{x^4}{64}$.

XXXIII.

(1.) $4ab^2$. (2.) $5ab$. (3.) $9xy$. (4.) $3ax$.

(5.) $7a^2x^2y$. (6.) $abuv$. (7.) $4a^2b$. (8.) $6x^2y^2$.

(9.) $4ab$. (10.) $5ab^2$. (11.) $2b^2$. (12.) $4uw$.

XXXIV.

(1.) $x+3$. (2.) $x-1$. (3.) $x-3$.

(4.) $x+3$. (5.) $2x-5$. (6.) $x-3$.

(7.) $x^2+10x+25$. (8.) x^2-5x+6. (9.) x^2-9.
(10.) $x-2$. (11.) $x-1$ (12.) x^2-3.
(13.) $2x+5$. (14.) $x-2$. (15.) x^2-2x+1.
(16.) $x-3y$. (17.) $x+y$. (18.) x^2+y.
(19.) $x+3y$. (20.) x^2-y^2.

XXXV.

(1.) $ax(2x+3)$. (2.) $2(x+3)$. (3.) $x(x-2)$.
(4.) $x-1$. (5.) $3x-2$.

XXXVI.

(1.) $a(x-a)$. (2.) $2a(x-3)$. (3.) $a(x-1)$.
(4.) $x+1$. (5.) $x+2$. (6.) $x-3$.

XXXVII.

(1.) $6abxy$. (2.) $24a^2x^2y$. (3.) $a^2b^2c^2$.
(4.) $24a^2b^2c^2$. (5.) $240a^2b^2c^2u^2v^2w^2$.
(6.) $(x^2-7+12)(x+2)=(x^2-x-6)(x-4)$.
(7.) $(2x^2-5x-3)(2x+1)=(4x^2+4x+1)(x-3)$.
(8.) $(3x^2-11x+6)(2x-1)=(2x^2-7x+3)(3x-2)$.
(9.) $(x^3-4ax^2+5a^2x-2a^3)(x^2+2ax+2a^2)=(x^3-2a^2x-4a^3)(x^2-2ax+a^2)$. (10.) $24(x+1)(x-1)^2$.
(11.) $(x-1)^2(x+1)^2$. (12.) $x^2y^2(x^2-y^2)$.
(13.) $a^3b(a^2-b^2)$. (14.) $12(x^2-1)(x^2+x+1)$.
(15.) $(p^2+q^2)(p^2-q^2)(p^2-pq+q^2)$.
(16.) $(p^4-1)(p^4+p^2+1)$. (17.) $(b-c)(c-a)(a-b)$.
(18.) $24a^2b^2(a^2-b^2)$.

XXXVIII.

(1.) $\frac{a}{3x}$. (2.) $\frac{bx}{cy}$. (3.) $\frac{3x}{4yz}$.
(4.) $\frac{5}{a-b}$. (5.) $\frac{2ab}{3b^2-6a^2}$. (6.) $\frac{1}{x-1}$.

(7.) $\frac{1}{x+2}$. (8.) $\frac{1}{x^3+1}$. (9.) $\frac{x+y}{x^2+xy+y^2}$.

(10.) $\frac{1}{x-1}$. (11.) $\frac{x-5}{x+1}$. (12.) $\frac{x+3}{x-5}$.

(13.) $\frac{3+x}{3-x}$. (14.) $\frac{x-10}{x^2-7x+10}$. (15.) $\frac{x-3}{x^2-3}$.

(16.) $\frac{3(4x-1)}{2(3x^2+1)}$. (17.) $\frac{x^2(x+2y)}{(x-2y)^2}$. (18.) $\frac{x-y}{x+y}$.

XXXIX.

(1.) $\frac{b}{ab}, \frac{2a}{ab}$. (2.) $\frac{b}{a^2b}, \frac{a}{a^2b}$. (3.) $\frac{2yz}{xyz}, \frac{3z}{xyz}, \frac{4}{xyz}$.

(4.) $\frac{2}{a}, \frac{-x}{a}$. (5.) $\frac{y}{axy}, \frac{-2a}{axy}, \frac{3}{axy}$.

(6.) $\frac{a(x+1)}{x^2-1}, \frac{2a}{x^2-1}$. (7.) $\frac{x+3}{(x+1)(x+3)}, \frac{2(x+1)}{(x+1)(x+8)}$.

(8.) $\frac{2}{x-1}, \frac{x-3}{x-1}$. (9.) $\frac{a}{a-b}, \frac{a-1}{a-b}$.

(10.) $\frac{4(x^2-1)}{x(x^2-1)}, \frac{3x(x-1)}{x(x^2-1)}, \frac{x}{x(x^2-1)}$.

(11.) $\frac{x^2-1}{x^3+1}, \frac{x^2-x+1}{x^3+1}, \frac{3x}{x^3+1}$.

(12.) $\frac{2(x-2)}{(x-1)(x-2)(x+2)}, \frac{3(x+2)}{(x-1)(x-2)(x+2)}, \frac{x-1}{(x-1)(x-2)(x+2)}$.

(13.) $\frac{c-b}{(b-c)(c-a)(a-b)}, \frac{a-c}{(b-c)(c-a)(a-b)}, \frac{b-a}{(b-c)(c-a)(a-b)}$.

(14.) $\frac{bx(x-b)}{abx(a-b)(x-a)(x-b)}, \frac{ax(a-x)}{abx(a-b)(x-a)(x-b)}, \frac{(a-b)(x-a)(x-b)}{abx(a-b)(x-a)(x-b)}$.

XL.

(1.) $\frac{a^2+b^2}{ab}$. (2.) $\frac{ax+2}{2a^2}$. (3.) $\frac{4a^2x+6a+6}{12a^2}$.

(4.) $\frac{3x-2}{3x^2}$. (5.) $\frac{8x^2+12ax-1}{4x^3}$.

(6.) $\frac{x^2-3x+6}{2x^2}$. (7.) $\frac{6b-17a}{60}$.

(8.) $\frac{2a^2+2b^2}{a^2-b^2}$. (9.) $\frac{4ab}{a^2-b^2}$. (10.) $\frac{1}{1-a}$.

(11.) $\frac{a+b}{a}$. (12.) $\frac{-2}{x(4x^2-1)}$. (13.) $\frac{2x^3}{x^4+x^2+1}$.

(14.) $\frac{1}{x^2(x^2-1)}$. (15.) $\frac{1}{x-2y}$. (16.) $\frac{2x+6}{x^4-1}$.

(17.) $\frac{2x+2y}{x^2+xy+y^2}$. (18.) $\frac{2x(2x^4+1)}{x^6-1}$.

(19.) $\frac{x-9}{(x-1)(x-2)(x+2)}$. (20.) $\frac{a^2}{(a-b)(a-c)}$.

(21.) $\frac{(a+b+c)x}{(x-a)(x-b)(x-c)}$. (22.) 0. (23.) 0.

XLI.

(1.) $\frac{3+5x}{x}$. (2.) $\frac{x-a^2}{a^2}$. (3.) $\frac{2a-x}{a}$.

(4.) $\frac{a+ax-x}{x}$. (5.) $\frac{3a-ax+x}{x}$.

(6.) $\frac{2x^2+x-1}{x^2}$. (7.) $\frac{2x^2+1}{x^2}$.

(8.) $\frac{4x^3-x^2+2x-3}{x^3}$. (9.) $\frac{1+3x}{x-1}$.

(10.) $\frac{a-a^2+ab}{a-b}$. (11.) $\frac{a+3b}{a+b}$.

(12.) $\frac{4x^2+9x-4}{x+3}$. (13.) $\frac{x^3-xy}{x-y}$.

(14.) $b+\frac{c}{a}$. (15.) $a-\frac{1}{x}$. (16.) $\frac{2}{3x}-1$.

(17.) $2x-1+\frac{3}{x}$. (18.) $\frac{4}{3x}-1+2x$.

(19.) $1+\frac{3x+5}{x^2-1}$. (20.) $5-\frac{1}{x-1}$.

(21.) $4-\frac{2x}{1+x}$. (22.) $x-3+\frac{3x-7}{x^2-3}$.

(23.) $3-\frac{x-2}{2x^2-x+1}$. (24.) $x^2-\frac{x^2+x-5}{x^3+1}$.

XLII.

(1.) $\frac{4a}{5x}$. (2.) 1. (3.) $\frac{a^3b^3c^3}{x^3y^3z^3}$.

(4.) $\frac{xy}{1-x^2}$. (5.) $\frac{a+b}{b}$. (6.) $\frac{a^2+ab+b^2}{a^2-ab+b^2}$.

(7.) $\frac{a^2+ab+b^2}{(a+b)^2}$. (8.) $\frac{(a+b)(a^2+ab+b^2)}{a^2b^2(a-b)}$.

(9.) $\frac{1}{x^3+1}$. (10.) $\frac{(x-1)^2}{(x^2-x+1)^2}$.

(11.) $yz+zx+xy$. (12.) $x-a$.

(13.) $\frac{a^4-b^4}{ab}$. (14.) $\frac{x}{x-y}$.

(15.) $\frac{b}{a(a^2-ab+b^2)}$. (16.) 1.

(17.) $\frac{xy}{x+y}$. (18.) $\frac{2}{(a-b)^2}$.

XLIII.

(1.) $\frac{4a^2b}{3x^2y}$. (2.) $\frac{4ax^2y}{3b}$. (3.) $\frac{x-1}{x+1}$.

(4.) $\frac{a-x}{a+x}$. (5.) $\frac{a^2+ab+b^2}{a^2-ab+b^2}$. (6.) $\frac{a^2+b^2}{a}$.

(7.) $\frac{a+x}{a}$. (8.) $\frac{2}{(a-b)^2}$. (9.) $\frac{a^2+p^2}{2ap}$.

(10.) $\frac{1}{a^2-x^2}$. (11.) $\frac{x^2+2xy-y^2}{x^2+y^2}$.

(12.) $\frac{ax}{a+x}$. (13.) $\frac{20}{99}$. (14.) $\frac{2}{99}$.

XLIV.

(1.) 10. (2.) 8. (3.) 12.

(4.) 6. (5.) 8. (6.) 2.

(7.) $-\frac{4}{5}$. (8.) -1. (9.) $-\frac{5}{9}$.

(10.) 20. (11.) $\frac{10}{3}$. (12.) 15.

(13.) $\frac{89}{49}$. (14.) $-\frac{3}{4}$. (15.) 3.

(16.) $\frac{9}{2}$. (17.) $\frac{abc}{b-a}$. (18.) $2.\frac{a^2+b^2}{a+b}$.

(19.) $a+b$. (20.) a. (21.) $\frac{1}{ab}$.

(22.) $\frac{a^2+b^2}{a+b}$.

XLV.

(1.) 40. (2.) 40. (3.) 355.

(4.) 4 bowled, 3 run out. (5.) $9\frac{1}{11}$. (6.) £5.

(7.) £9150 of 3 per cents., £5820 of $3\frac{1}{4}$ per cents.

(8.) A £1250, B £1500. (9.) 100.

(10.) 7 miles an hour. (11.) 45 and 30 miles an hour.

(12.) C 42 days, B 84 days, A 168 days.

(13.) 240, 180, 144 days. (14.) $5\frac{5}{11}$ minutes past 7.

(15.) $26\frac{8}{11}$ minutes past 2. (16.) $49\frac{1}{11}$ minutes past 3.

(17.) $32\frac{8}{11}$ minutes past 3. (18.) 10 gallons.

(19.) 30 gallons. (20.) 112 oz.

(21.) 70 grains. (22.) $2\frac{3}{4}$ oz.

(23.) 3 miles an hour. (24.) $4\frac{1}{2}$ miles.

(25.) 10 yards in 200. (26.) 1430 yards.

(27.) 1 mile; half-an-hour. (28.) $5\frac{2}{3}$.

(29.) $\frac{2}{95}$ of a mile behind.

XLVI.

(1.) 6, −6. (2.) 3, −3. (3.) 9, −9.
(4.) 4, −4. (5.) 2, −2. (6.) $\sqrt{26}$, $-\sqrt{26}$.
(7.) 0, 3. (8.) 0, −12. (9.) 0, 19.
(10.) 0, $\frac{7}{2}$. (11.) 0, $-\frac{1}{12}$. (12.) 0, $-\frac{5}{3}$.
(13.) 3, 5. (14.) −5, 7. (15.) −1, −3.
(16.) $\frac{1}{2}$, $\frac{4}{3}$. (17.) $\frac{5}{3}$, $-\frac{7}{2}$. (18.) $-\frac{6}{5}$, $-\frac{7}{6}$.
(19.) $\frac{b}{a}$, $-\frac{d}{c}$. (20.) 0, a.

XLVII.

(1.) 2, 4. (2.) 5, −1. (3.) 3, −7.
(4.) 2, $\frac{1}{2}$. (5.) $\frac{1}{2}$, −2. (6.) $\frac{1}{2}$, $\frac{1}{4}$.
(7.) $\frac{5}{2}$, $-\frac{3}{2}$. (8.) $\frac{1}{2}$, $\frac{4}{3}$. (9.) 6, −4.
(10.) $\frac{2}{3}$, $\frac{3}{2}$. (11.) 2, $\frac{1}{2}$. (12.) $\frac{43}{5}$, −10.
(13.) 2, $-\frac{2}{15}$. (14.) 14, −10. (15.) 4, −1.
(16.) −1, $-\frac{5}{2}$. (17.) 4, 4. (18.) 1, $\frac{13}{3}$.
(19.) 1, 2. (20.) 3, $-\frac{4}{5}$.
(21.) 6, $\frac{9}{2}$. (22.) a, b.

XLVIII.

(1.) 60 ft. by 30 ft. (2.) 13 and 7.
(3.) 2 or 10. (4.) 12 and 3.
(5.) 4 or 6. (6.) 5, 4, 3; or −1, 0, 1.
(7.) 30 ft. (8.) £20. (9.) £90 or £10.
(10.) A's rate 4 miles; B's $2\frac{1}{2}$ miles an hour.
(11.) $\frac{2}{5}$. (12.) 10. (13.) 87.
(14.) 2 ft. (15.) 11. (16.) 23.
(17.) 12. (18.) 56. (19.) 1296.

XLIX.

(1.) 2, 3. (2.) 3, 5. (3.) 5, 4.
(4.) 6, 12. (5.) 30, 12. (6.) 4, 3.
(7.) 4, -1. (8.) 3, 5. (9.) 2, 3.
(10.) 2, 3. (11.) 9, 4. (12.) 60, 40.
(13.) 13, 3. (14.) 4, 1. (15.) $\frac{a+b}{2}$, $\frac{a-b}{2}$.
(16.) a, $\frac{b^2}{a}$. (17.) $\frac{ab-ac}{b-a}$, $\frac{ab-bc}{a-b}$.

L.

(1.) 2, 1, 3. (2.) 1, 3, 2. (3.) 5, 4, 3.
(4.) 3, 4, -3. (5.) 3, 2, 1. (6.) 4, 5, 6.
(7.) 4, 3, 2. (8.) 19, 7, 4. (9.) 2, 4, 7.
(10.) $\frac{b+c-a}{2}$, $\frac{c+a-b}{2}$, $\frac{a+b-c}{2}$.

LI.

(1.) 72. (2.) 378 and 216.
(3.) 5 and 4. (4.) $\frac{13}{5}$. (5.) 84.
(6.) A to B 37 miles, B to C 45 miles, C to A 52 miles.
(7.) 19 five-franc pieces, 11 two-franc pieces.
(8.) 296 for A, 1488 for B, 196 for C.
(9.) The stream 3 miles an hour; the boat 8 miles an hour.
(10.) 148 for A, 750 for C and A, 158 for A and B.
(11.) The gold coins are half-sovereigns, the silver coins are crowns.
(12.) A to B $11\frac{1}{8}$ miles, B to C 7 miles, C to D $5\frac{1}{8}$ miles.
(13.) 24 octavos or 32 duodecimos.
(14.) A thaler = 2*s.* 11*d.*; a franc = $9\frac{1}{2}$*d.*; a florin = 1*s.* $11\frac{1}{2}$*d.*

(15.) A Prussian pound $=16\frac{1}{2}$ oz.; an Austrian pound $=$ $19\frac{3}{4}$ oz.; a kilogramme $= 35\frac{1}{4}$ oz.

(16.) $3\frac{1}{2}$ miles.

LII.

(1.) $\sqrt[3]{a}$, $\sqrt[6]{a}$, $\sqrt[5]{a^2}$, $\sqrt{a^3}$. (2.) $\frac{1}{x^2}$, $\frac{1}{x^8}$, $\frac{1}{x^{10}}$.

(3.) $\frac{1}{\sqrt[3]{m}}$, $\frac{1}{\sqrt{n^5}}$, $\frac{1}{\sqrt[4]{p^7}}$. (4.) $2\sqrt[4]{a}$, $\frac{3}{x^2}$, $\frac{6}{\sqrt{m^3}}$.

(5.) $x^{\frac{1}{2}}$, $m^{\frac{1}{5}}$, $n^{\frac{1}{8}}$. (6.) x^{-1}, a^{-2}, a^{-5}, a^{-8}.

(7.) $x^{\frac{3}{2}}$, $x^{\frac{2}{3}}$, $x^{\frac{4}{5}}$. (8.) $x^{-\frac{1}{2}}$, $x^{-\frac{1}{3}}$, $x^{-\frac{1}{5}}$.

(9.) $2m^{-1}$, $3n^{-2}$, $10p^{-3}$. (10.) $2x^{-\frac{1}{2}}$, $5x^{-\frac{1}{3}}$, $7x^{-\frac{2}{5}}$.

LIII.

(1.) $6x^{n+1}$; $4x^{m+2}$; $4x^{3m}$. (2.) $2x^{\frac{3}{4}}$; $6x^{\frac{5}{6}}$; $30x^{\frac{5}{4}}$.

(3.) $x^{\frac{2n+1}{2}}$; $6x^{\frac{3n}{2}}$; $x^{\frac{5n}{6}}$. (4.) $2a$; $3a^{-1}$; $30a^{-1}$.

(5.) $a^{\frac{1}{6}}$; $2a^{-\frac{1}{12}}$; $a^{\frac{2}{3}}$. (6.) $a^{\frac{p}{6}}$; $a^{\frac{n}{2}}$; $a^{\frac{5n}{3}}$.

(7.) 1; 1; 6; mn.

LIV.

(1.) a^m; a^{2n}. (2.) $a^{\frac{1}{3}}$; $a^{\frac{1}{3}}$. (3.) $x^{\frac{1}{2}}$; x.

(4.) x^3; x^5. (5.) x; x^3. (6.) x; $x^{\frac{5}{2}}$.

(7.) x^n; $x^{\frac{3n}{2}}$.

LV.

(1.) a^{12}; a^8; a^9. (2.) a^{-2}; a^{-6}; a^{-12}.

(3.) a^{-6}; a^6; a^{12}. (4.) $a^{\frac{5}{2}}$; a^9; a^{10}.

(5.) a; $a^{\frac{1}{2}}$; $a^{\frac{5}{3}}$. (6.) $a^{\frac{2}{3}}$; $a^{\frac{4}{3}}$; $a^{\frac{2}{3}}$.